Math 163 Midterm 3 Review Sheet. Sample problem solutions.

1. One possible example is $\mathbf{r}(t) = \cos t\mathbf{i} + \cos t\mathbf{j} + \cos t\mathbf{k}$.

2. (The function gives a parametrization of the parabola $y = x^2$ and the point where $t = 0$ is the origin. You don't need to know that to answer the question, I suppose.) $\mathbf{r}'(t) = \langle 1, 2t, 0\rangle$, so $\mathbf{T}(t) = \dfrac{\mathbf{i}}{\sqrt{1+4t^2}} + \dfrac{2t\mathbf{j}}{\sqrt{1+4t^2}}$. Thus $\mathbf{T}(0) = \mathbf{i}$ and you can compute that $\mathbf{N}(0) = \mathbf{j}$ – this is easily done by recalling that $\mathbf{N}$ should be a unit vector orthogonal to $\mathbf{T}$ pointing 'into' the curve, in the same 'plane' as $\mathbf{T}$. Or you can compute $\mathbf{T}'(t)$ directly...

3. (a) $\mathbf{v}(t) = \left\langle \frac{3}{2}e^{t/2}, -2e^{-t/2}\right\rangle$, $\mathbf{a}(t) = \left\langle \frac{3}{4}e^{3/2}, e^{-t/2}\right\rangle$.
   (b) They are parallel – in fact, the acceleration vector is 1/4 times the position vector
   (c) $\mathbf{v}(0) = \left\langle \frac{3}{2}, -2\right\rangle$ which is not perpendicular to $\mathbf{a}(0) = \left\langle \frac{3}{4}, 0\right\rangle$, so the unit tangent vector isn't orthogonal either.

4. (a) We just compare the z-coordinates at $t = 0$ and $t = 10$ – the total change in altitude is $10\sqrt{2}$ hectometers. (that's around 1.5 km).
   (b) $\mathbf{r}'(t) = \langle e^t, -e^{-t}, \sqrt{2}\rangle$, so the speed in the x-direction is e^4 hectometers/hr.
   (c) e^{-4} hm/hr
   (d) We must compute $\int_0^4 |\mathbf{r}'(t)|\, dt$, the arclength of the curve. Note that $|\mathbf{r}'(t)| = \sqrt{e^2t + e^{-2t} + 2} = \sqrt{(e^t + e^{-t})^2} = (e^t + e^{-t})$. The total distance traveled is $e^4 - e^{-4}$ hm.

5. (a) $\mathbf{v}(t) = \langle 2\cos 2t, 3, -2\sin 2t\rangle$ and $\mathbf{a}(t) = \langle -4\sin 2t, 0, -4\cos 2t\rangle$. Their dot-product does indeed equal zero for all t, so they are orthogonal for all t.
   (b) $\mathbf{r}\cdot\mathbf{v} = 2\sin 2t\cos 2t + 9t - 2\sin 2t\cos 2t = 0$ when $t = 0$.

6. $\mathbf{r}'(t) = \langle 3t^2 - 3, 2t\rangle$ so $\mathbf{T}(t) = \left\langle \dfrac{3t^2-3}{\sqrt{9t^4-14t^2+9}}, \dfrac{2t}{\sqrt{9t^4-14t^2+9}}\right\rangle$. Now, when does the particle cross the y-axis? When the x-coordinate is zero: that happens when $t^3 - 3t = 0$, which has three solutions: $t = 0, \pm\sqrt{3}$. The unit tangent vector when $t = 0$ is $\langle -1, 0\rangle$. The unit tangent vector when $t = -\sqrt{3}$ is $\left\langle \dfrac{6}{\sqrt{48}}, \dfrac{-\sqrt{3}}{24}\right\rangle = \left\langle \dfrac{\sqrt{3}}{2}, \dfrac{-\sqrt{3}}{24}\right\rangle$. The unit tangent vector when $t = \sqrt{3}$ is $\left\langle \dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{3}}{24}\right\rangle$.

7. $\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$

8. $\frac{4}{3}\mathbf{i} + \frac{1}{4}\mathbf{j}$

9. It is $\big(t\cos t - \sin t - t^2\sin t\big)\,\mathbf{i} + (4t + \sin t)\,\mathbf{j} + \big(6t^2 - \cos t\big)\,\mathbf{k}$. You can get this either by doing the cross-product first, then taking derivatives, or using the appropriate derivative rule from section 13.2.

10. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = 0$ (it is easiest to do this by taking the dot product first, before taking a derivative!) and $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{u}(t)] = \sin^2 t + 2t\sin t\cos t + wt + \frac{4}{3}t^{1/3}$.

11. $2\sqrt{26}$

12. If $\mathbf{r}'$ and $\mathbf{r}''$ are parallel, then their cross-product must be zero, then thus their curvature must also be zero. An example of a curve where the velocity and acceleration are always parallel will therefore have to be a curve that is a straight line – for example $\langle t, t, t \rangle$, or even $\langle t^2, 2t^2, 3t^2 \rangle$. (Note that technically the zero vector $\mathbf{0}$ is both orthogonal to any vector, *and* parallel to any vector!)

13. The position vector of the cannonball is $\mathbf{r}(t) = 80\cos 30° t\mathbf{i} + \left(264 + 80\sin 30° t - 16t^2\right)\mathbf{j}$. We are going to have to find the value of t for which the y-component of the position vector is 0 – this means solving the quadratic equation $-16t^2 + 40t + 264 = 0$. Dividing both sides by -8 we get $2t^2 - 5t - 33 = 0$, which we can solve by factoring to get $t = -3, 11/2$. We ignore the negative value of t – we see that the cannonball will spend 5.5 seconds in flight. From there we can find the x-coordinate when t=11/2: $x = 80 \cdot \frac{\sqrt{3}}{2} \cdot \frac{11}{2} = 220\sqrt{3}$ feet. So the cannonball lands $220\sqrt{3} \approx 381$ feet from the base of the tower.

    To find its speed at that moment, we consider $\mathbf{v}(t) = 40\sqrt{3}\mathbf{i} + (40 - 32t)\mathbf{j}$ and plug in $t = 5.5$ to get $40\sqrt{3}\mathbf{i} + (-136)\mathbf{j}$, the magnitude of which is about 152.63, so the cannonball is going about 152.63 feet/sec at the moment it hits the ground.

    Now we consider the maximum height – I will simply use the classic formula for max height assuming launch point is the origin, $y_{max} = \dfrac{(v_0 \sin\alpha)^2}{2g} = \dfrac{(80 \cdot 0.5)^2}{64} = 25$, then add 264 feet to that, to get a maximum height of 289 feet.

14. The domain consists of all points on, or to the right of, the horizontal parabola $x = y^2$. The range is $[0, \infty)$.

15. They are all circles centered at the point $(-1.5, 2)$.

16. We can easily determine that (2,3) is on the curve $x^2 + 2y = 10$, an equation which (4, -3) also clearly satisfies.

17. 2

18. 6

19. -2

20. 0

21. $f_x = y(\ln z)z^{xy}$, $f_y = x(\ln z)z^{xy}$. $f_z = xyz^{xy-1}$. $f_y(2, 1, e) = 2e^2$.

22. $\dfrac{\partial f}{\partial x} = \dfrac{5y}{(3x - 2y)^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{-5x}{(3x - 2y)^2}$

23. $f_{xy} = \frac{1}{y}$, $f_{xz} = \frac{2}{z}$, $f_{yz} = 0$.

24. $f_x = g'\left(x^2 + y^2\right)2x$ (by the chain rule) and $f_y = g'\left(x^2 + y^2\right)2y$, so then $yf_x - xf_y = 2xyg'\left(x^2 + y^2\right) - 2xyg'\left(x^2 + y^2\right) = 0$.

25. (a) The graph of $f(x, y) = \frac{1}{5}$ is a circle of radius 2 centered at the origin; the other level curve is a circle of radius centered at the origin.
    (b) The graph of $f(x, y) = 1$ consists of a single point: the origin.

26. Yes.

27. Yes – for example, $f(x, y) = e^x + e^{-y}$. Any function of the form $f(x, y) = g(x) + h(y)$, where $g'(x) > 0$, $h'(y) < 0$, and $g + h > 0$ will work...