Math 163 Midterm 2 Review Sheet. Sample problem solutions.

1. One way to do this is just to set up the equation $\sqrt{x^2+y^2+z^2} = \sqrt{(x-2)^2+(y-4)^2+(z-8)^2}$, representing how a point (x, y, z) is the same distance from the origin as it is from (2,4,8). Then square both sides and simplify. OR, you could use the hint given. The plane must be perpendicular to the line connecting the two points, and the plane must go through the point (1,2,4) (which is the midpoint of the line segment connecting the points), and so an equation must be $2x+4y+8z=42$.

2. True or false?
   (a) Although this is true for a few special cases, this is not true for all pairs of vectors. Consider $\langle 1,0,0\rangle$ and $\langle 0,1,0\rangle$, for example.
   (b) This is false – the right-hand side of the equation is literally meaningless. We have not defined what it means to divide one vector by another.
   (c) Although this is true in some cases, it is not true generally – for example, if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then the scalar projections will be zero, but the lengths could be very different.
   (d) This is true. All three vectors are orthogonal to each other – we know this because their dot products are all zero. Therefore the cross-product of two of the vectors MUST be parallel to the third vector, because the cross-product of two vectors must be orthogonal to both vectors.

3. I get $\langle -4/3, 16/3, -8/3\rangle$.

4. It is a sphere of radius 7 centered at (6, 3, -2).

5. Start by taking the cross product of the two vectors to get a vector that is orthogonal to both of them. The cross product of the first with the second is $\langle -6,3,1\rangle$ which has length $\sqrt{41}$. So therefore $\pm\frac{2}{\sqrt{41}}\langle -6,3,1\rangle$ are the two vectors we're looking for.

6. One possible answer is $\mathbf{r}(t) = \langle -2,4,10\rangle + t\,\langle 6,-4,-14\rangle$.

7. One possible answer is $6x-4y-14z+44=0$.

8. We know that $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$ so work out the value of that fraction and do inverse cosine and ... I get about 165 degrees.

9. Find the direction angles of $\mathbf{v} = \langle 3,-2,5\rangle$. $\alpha = \cos^{-1}\left(\frac{3}{\sqrt{38}}\right) \approx 61°$. Similarly, $\beta \approx 109°$ and $\gamma \approx 36°$.

10. $\lim_{t\to\infty} \mathbf{r}(t) = \langle -1,0,0\rangle$. (I had to use the Squeeze Theorem – for functions, not sequences, technically – in the second component.)

11. (a) We compute $|\mathbf{u}\times\mathbf{v}| = |\,\langle 2,-3,1\rangle\,| = \sqrt{14}$.
    (b) Find the area of the parallelopiped determined by $\mathbf{u},\mathbf{v},\mathbf{w}$. We compute the magnitude of the box product, $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\,| = |\,\langle 1,1,-1\rangle\cdot\langle 5,-8,-2\rangle\,| = 1$. The volume is 1. That is smaller than I was expecting. Huh.

12. (a) For this we compute half the area of the parallelogram determined by $\overline{AB}$ and $\overline{AC}$. $\langle 1,-4,0\rangle \times \langle 2,1,-8\rangle \approx 34.2$, so we divide that by two to get about 17.1.

(b) Happily we just computed a cross-product in the last part, which we need for the normal vector to the plane. (The normal vector is $\langle 32, 8, 9\rangle$.) So for my plane equation I'm going to go with $32x + 8(y-2) + 9(z-3) = 0$. If you want to simplify that to $32x + 8y + 9z - 43 = 0$, go right ahead.

(c) Find parametric equations for the line where the plane containing the triangle intersects the plane $2x - 2y - z - 3 = 0$. There are many possible answers to this. First note that the direction of the line should be orthogonal to the normal vectors for both planes. So let's find the cross product of the normal vectors: $\langle 32, 8, 9\rangle \times \langle 2, -2, -1\rangle = \langle 10, 50, -80\rangle$. So that gives the direction of our line. So does $\langle 1, 5, -8\rangle$. Now we just need a point on the line. You can do this by picking a value for x, y, or z and plugging in that value into both plane equations, then solving the resulting system of two equations in two unknowns in order to find a point of intersection with the chosen value. Letting $x = 0$ gives $z = 11$ and $y = -7$. So the point $(0, -7, 11)$ lies on both planes, and therefore the line, so a parametrization of the line would be $x = t,\ y = -7 + 5t,\ z = 11 - 8t$. (You may be amused to note that for this given parametrization, $t = 1$ gives point B and $t = 2$ gives point C. If we had noticed that either of those points satisfied the equation $2x - 2y - z - 3 = 0$ we could have skipped solving the system of equations. But that won't always happen in this kind of problem.)

13. First we find a parametrization for the line: $x = 2t,\ y = -t,\ z = -t$ will work. Now we try to find where it intersects the plane $3x - 5y + 2z = 6$. To do this we will first find the value of t for which this intersection occurs. We solve $3(2t) - 5(-t) + 2(-t) = 6$ to get $t = 2/3$ and from there we see that the point of intersection must be $(4/3, -2/3, -2/3)$.

14. Here's *a* way to do this (there may be simpler ways but this is what I've got): first we find a normal vector to the plane, $\langle 1, -1, 0\rangle$. Then we find a line going through the point (6, 0, -6) that is parallel to that normal vector: $x = 6 + t,\ y = -t,\ z = -6$. Then we find where that line intersects the plane: that happens when $t = -1$, at the point (5, 1, -6). Now we find the distance between that point and (6, 0, -6), which happens to be $\sqrt{2}$ – and that's our answer.

15. One possible answer is $\mathbf{r}(t) = \left\langle -3 + 3\cos\left(\frac{\pi t}{2}\right), -4 + 3\sin\left(\frac{\pi t}{2}\right), 8\right\rangle$. (Remember how sinusoidal functions work!)