Math& 163 Projectile Motion Questions

For all the questions below, unless otherwise noted, assume ideal projectile motion. I do not expect anyone to finish all these problems in one class session!

1. A golf ball leaves the ground at a thirty-degree angle at a speed of 90 ft/sec. Will it clear the top of a thirty-five-foot tree that is in the way, 135 feet down the fairway?

2. A human cannonball is to be fired with an initial speed of $\frac{80\sqrt{10}}{3}$ ft/sec. The performer is supposed to land in a special net located 200 feet downrange at the same height as the muzzle of the cannon. The circus is being held in a large room with a flat ceiling 75 feet higher than the muzzle. Can the performer be fired to the cushion without striking the ceiling? If so, what should the cannon's angle of elevation be?

3. Show that a projectile fired at an angle of α degrees, $0 < \alpha < 90$, has the same range as a projectile fired at the same speed at an angle of $(90 - \alpha)$ degrees. (If air resistance were taken into account, this would not be true.)

4. One type of model of projectile motion that takes into account air resistance does so by assuming the 'drag force' due to air resistance is a force acting opposite the velocity vector, with magnitude proportional to the speed of the projectile. In this model, the position vector of a projectile launched from the origin is

$$\mathbf{r}(t) = \left[\frac{v_0}{k}\left(1 - e^{-kt}\right)\cos\alpha\right]\mathbf{i} + \left[\frac{v_0}{k}\left(1 - e^{-kt}\right)\sin\alpha + \frac{g}{k^2}\left(1 - kt - e^{-kt}\right)\right]\mathbf{j}$$

   where k is the 'drag coefficient', a positive constant representing resistance due to air density.

   (a) Show that the vector function above satisfies $\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j} - k\mathbf{v}$. (This shows that the forces acting on the object are gravity and a force proportional to the velocity.)

   (b) Ichiro hits a baseball when it is 3 feet above the ground, leaving the bat with initial speed 152 ft/sec, making an angle of 20 degrees with the horizontal. (This is the same set-up as a problem done in class, but without a gust of wind.) Assume a linear drag caused by air resistance (so you have to use the formula above) with a drag coefficient $k = 0.12$. A 10-foot-high outfield fence is 340 feet from home plate in the direction of the flight of the baseball. The outfielder can jump and catch any ball up to 11 feet off the ground to stop it from going over the fence. Will the ball make it over the fence?

Solutions

1. We place the origin of a coordinate system at the ball's launching point. We have to determine the y-coordinate on the graph of its trajectory when the x-coordinate is 135. The vector equation for its trajectory is

$$\mathbf{r}(t) = [90\cos(30^\circ)t]\,\mathbf{i} + \left[90\sin(30^\circ)t - 16t^2\right]\mathbf{j} = 45\sqrt{3}t\mathbf{i} + \left[45t - 16t^2\right]\mathbf{j}$$

and so first we solve $45\sqrt{3}t = 135$ to get $t = \sqrt{3}$. Then the y-coordinate for that value of t is $45\sqrt{3} - 48$, which is 29.9 feet or so. It looks like the ball will not clear the tree.

2. Again we place the origin at the muzzle of the cannon. Let α be the launch angle. We can then write the trajectory of the human cannonball as

$$\mathbf{r}(t) = \left[\frac{80\sqrt{10}}{3}\cos\alpha\; t\right]\mathbf{i} + \left[\frac{80\sqrt{10}}{3}\sin\alpha\; t - 16t^2\right]\mathbf{j}$$

and we must also employ the formula for the range of a projectile,

$$x_{max} = \frac{v_0^2}{g}\sin 2\alpha$$

to determine that to acheive a range of 200 feet, $200 = \frac{\frac{64000}{9}}{32}\cdot\sin 2\alpha$ and therefore $\sin 2\alpha = 0.9$ and thus that $2\alpha \approx 64^\circ$, so $\alpha \approx 32^\circ$. (While it could also be true that $2\alpha \approx 116^\circ$ and therefore $\alpha \approx 58^\circ$, we will ignore this, since a higher angle would lead to an increased likelihood of crashing into the ceiling.) So then we consult our formula for the max height,

$$y_{min} = \frac{v_0^2}{2g}\sin\alpha$$

and determine that for $\alpha \approx 32^\circ$, the max. height is well under 75 feet. So, yes, the human cannonball can make it, and the angle of elevation should be about 32 degrees.

3. For $0 < \alpha < 90$, note that $\sin[2\cdot(90-\alpha)] = \sin[180 - 2\alpha] = \sin[2\alpha]$, so therefore the ranges will be the same.

4. We determine that

$$\mathbf{v}(t) = v_0\cos\alpha e^{-kt}\mathbf{i} + \left(v_0\sin\alpha e^{-kt} - \frac{g}{k} + \frac{g}{k^2}e^{-kt}\right)\mathbf{j}$$

and then...

$$\begin{aligned}
\mathbf{r}(t) &= -kv_0\cos\alpha e^{-kt}\mathbf{i} - kv_0\sin\alpha e^{-kt}\mathbf{j} - \frac{g}{k}e^{-kt}\mathbf{j} \\
&= -g\mathbf{j} - kv_0\cos\alpha e^{-kt}\mathbf{i} - kv_0\sin\alpha e^{-kt}\mathbf{j} + g\mathbf{j} - \frac{g}{k}e^{-kt}\mathbf{j} \\
&= -g\mathbf{j} - k\left[v_0\cos\alpha e^{-kt}\mathbf{i} + v_0\sin\alpha e^{-kt}\mathbf{j} - \frac{g}{k}\mathbf{j} + \frac{g}{k^2}e^{-kt}\mathbf{j}\right] \\
&= -g\mathbf{j} - k\cdot\mathbf{v}(t)
\end{aligned}$$

Whew! Then... we have to find the y-coordinate when the x-coordinate is 340. We must solve

$$340 = \frac{152}{0.12}\left(1 - e^{-0.12t}\right)\cos(20^\circ)$$

The solution is $t = \dfrac{\ln\left(1 - \frac{340(0.12)}{152\cos(20^\circ)}\right)}{-0.12} \approx 2.8$ seconds. The y-coordinate for that value of t is 11.01. Which means that it will JUST BARELY make it over the outfielder's mitt and into the stands for a home run. Go Mariners!

(You get 11.01359495 if you plug in t=2.803158717 with no rounding anywhere. If you just plug in 2.8, you get 11.14 feet, which seems like a slightly more clear-cut home run. If you rounded off some numbers in the middle of solving for t above, I just don't know what happens. So notice how choosing where to round makes a big difference. In math classes we get to believe in 'exact values' and perfect numbers, but in chemistry or physics or other hard sciences, always pay attention to your science professors when they give your directions for how to round your answers. Ichiro is counting on you!)