

Differential Equations - Practice Problem Answers for the Last Three Weeks of Material

1. (a) Displacement function: $y = -2 \sin(2t) + \frac{1}{2} \cos(2t)$

$$\text{Amplitude} = \sqrt{(-2)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{17}}{2} \text{ meters}$$

$$\text{Frequency} = \frac{2}{2\pi} = \frac{1}{\pi} \text{ cycles/sec}$$

- (b) Solving $y = 0$: $t = \frac{1}{2} \arctan(0.25) \approx 0.1225 \text{ sec}$

(c) $l = \frac{mg}{k} = \frac{9.81}{4} = \boxed{2.45 \text{ m}}$

- (d) Since the system is undamped, the resonance frequency is $\frac{\gamma_r}{2\pi}$ with $\gamma_r = \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2$.

So, the resonance frequency is $\frac{1}{\pi}$ cycles/second, which is the same as the frequency of the displacement function.

2. (a) Displacement function: $y = \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}$

Solving $y'(t) = 0$: $t = \frac{\ln(4)}{3} \approx 0.4621 \text{ seconds}$

- (b) Steady-State Solution: $y_p = \frac{3}{25} \sin(3t) - \frac{1}{25} \cos(3t)$

OR $y_p = \frac{\sqrt{10}}{25} \sin(3t + \phi)$, with $\phi = \arctan(-\frac{1}{3})$

3. $x(t) = c_1 e^t + c_2 t e^t + \frac{1}{2} e^{-t}$ and $y(t) = -c_1 e^t + \frac{1}{2} c_2 e^t - c_2 t e^t - 2e^{-t}$

OR

$$x(t) = -(c_1 - \frac{1}{2}c_2)e^t - c_2 t e^t + \frac{1}{2}e^{-t} \quad \text{and} \quad y(t) = c_1 e^t + c_2 t e^t - 2e^{-t}$$

4. $x(t) = 4.5e^t + 9te^t + \frac{1}{2}e^{-t}$ and $y(t) = -9te^t - 2e^{-t}$

5. Let $A(t)$ = temperature of room A at t hours, $B(t)$ = temperature of room B at t hours.

System: $\frac{dA}{dt} = \frac{1}{4}[100 - A] + \frac{1}{2}[B - A] - 7$, $\frac{dB}{dt} = 1[100 - B] + \frac{1}{2}[A - B]$

Simplified: $A' = 18 - 0.75A + 0.5B$, $B' = 100 - 1.5B + 0.5A$

Solution for $B(t)$: $B(t) = c_1 e^{-t/2} + c_2 e^{-7t/4} + 96$

So, as time goes on, room B approaches $\boxed{96^\circ}$.

6. (a) Phase-plane equation: $\frac{dy}{dx} = \frac{y}{2x}$

Solution (Integral Curves): $y^2 = Cx$ OR $x = C^*y^2$

(b) The trajectories should flow to the right when $y > 0$ and to the left when $y < 0$.

