Math 207
Final Exam Answers

1. (a) This equation is linear. After dividing by x^2, we can find an integrating factor is $\mu(x) = x^4$.
   Solution: $\boxed{y = \frac{3}{2}x^2 + x^{-1} + Cx^{-4}}$

   (b) This is a separable equation. $\Rightarrow \quad \int y^{-2}\,dy = \int x^2 \sec^2(x^3+1)\,dx$

   Using substitution on the right side of the equation: $\quad -y^{-1} = \frac{1}{3}\tan(x^3+1) + C$

   Solution: $\boxed{y = \dfrac{1}{-\frac{1}{3}\tan(x^3+1) - C} \text{ and the lost solution } y \equiv 0}$

   (c) This is not exact, but you can use a special integrating factor to make it exact.

   Since $\dfrac{\partial N/\partial x - \partial M/\partial y}{M} = \dfrac{3}{y}$, we have a special integrating factor of $\mu(y) = y^3$.

   The new equation $\quad 2x^7y^4\,dx + (x^8y^3 + 15y^2)dy = 0 \quad$ is exact.

   $F(x,y) = \frac{1}{4}x^8y^4 + 5y^3 \quad \Rightarrow \quad$ Solution: $\boxed{\frac{1}{4}x^8y^4 + 5y^3 = C}$

2. (a) Differential Equation: $\dfrac{dT}{dt} = \frac{1}{2}[100 - T] - 8 \qquad$ Initial Condition: $\quad T(0) = 85$

   This is a linear and separable equation. The solution with the initial condition is $\boxed{T = 84 + e^{-t/2}.}$

   (b) $\boxed{\text{No}}$ (If you try to solve $T = 90$, you will get a negative value for time because at time 0, the room is 85°F and it cools down to 84°F.)

3. You can use either substitution or elimination.

   - If you solve for x first, you will get the differential equation: $\quad x'' - 2x' + x = -t, \quad$ which gives $\boxed{x = c_1e^t + c_2te^t - t - 2.}$

     Using equation 1 to get y: $\quad \boxed{y = (2c_1 - c_2)e^t + 2c_2te^t - 3t - 5}$

   - If you solve for y first, you will get the differential equation: $y'' - 2y' + y = -3t + 1, \quad$ which gives $\boxed{y = c_1e^t + c_2te^t - 3t - 5.}$

     Using equation 2 to get x: $\quad \boxed{x = (\frac{1}{2}c_1 + \frac{1}{4}c_2)e^t + \frac{1}{2}c_2te^t - t - 2}$

4. (a) Differential Equation: $y'' + 6y' + 25y = 0$ Initial Conditions: $y(0) = 8,\quad y'(0) = 0$

General Solution: $y = c_1 e^{-3t}\cos(4t) + c_2 e^{-3t}\sin(4t)$

Using the initial conditions: $\boxed{y = 8e^{-3t}\cos(4t) + 6e^{-3t}\sin(4t)}$

(b) Solving $8e^{-3t}\cos(4t) + 6e^{-3t}\sin(4t) = 0$ $t = \frac{1}{4}\arctan(-\frac{4}{3}) + \frac{\pi}{4}(n)$ for some integer n

So, the first positive time is $\boxed{\frac{1}{4}\arctan(-\frac{4}{3}) + \frac{\pi}{4} \approx 0.5536 \text{ seconds}}$

5. (a) $\frac{dx}{dt} = 0$ if $x = 1$, $\frac{dy}{dt} = 0$ if $y = 0$

So, the equibrium solution for the system is the point $\boxed{(1,0).}$

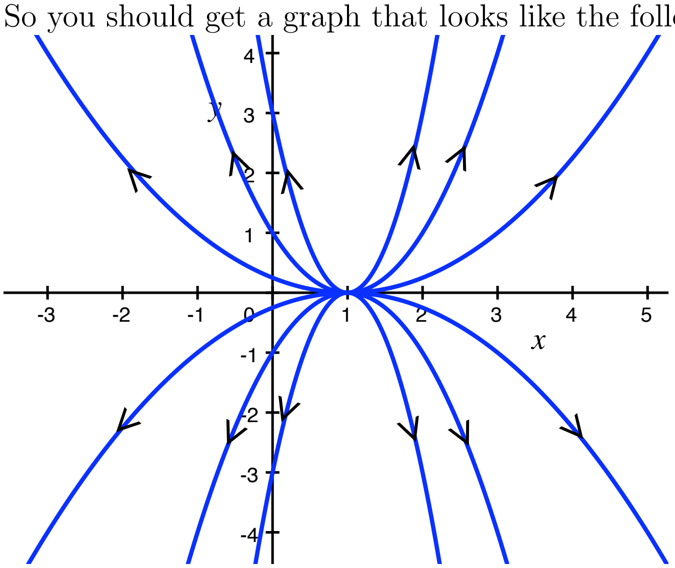
(b) Phase plane equation: $\frac{dy}{dx} = \frac{2y}{x-1}$ This is separable.

Solution: $y = C(x-1)^2$ for any constant C

These integral curves are parabolas with a vertex at (1,0).

You can use either $\frac{dx}{dt}$ or $\frac{dy}{dt}$ to find the direction of flow (trajectories as $t \to \infty$).
If you use $\frac{dx}{dt}$, then since $\frac{dx}{dt} > 0$ when $x > 1$, we have that solutions flow to the right whenever $x > 1$.
Since $\frac{dx}{dt} < 0$ when $x < 1$, we have that solutions flow to the left whenever $x < 1$.

So you should get a graph that looks like the following:



(c) If you start at the point $(-2, 4)$, then looking at the direction of flow, we can see that $\boxed{x(t) \to -\infty \text{ and } y(t) \to \infty.}$

6. (a) A = Amount of salt in tank A (in kg) at t minutes
B = Amount of salt in tank B (in kg) at t minutes

Differential Equations:

$$\boxed{\frac{dA}{dt} = 0.6 + 2(\frac{B}{40}) - 5(\frac{A}{40}) \qquad \text{or} \quad A' = 0.6 + 0.05B - 0.125A}$$

$$\boxed{\frac{dB}{dt} = 5(\frac{A}{40}) - 5(\frac{B}{40}) \qquad \text{or} \quad B' = 0.125A - 0.125B}$$

Initial Conditions: $\boxed{A(0) = B(0) = 0}$

(b) As time goes on, the concentration of each tank will approach $\boxed{0.2 \text{ kg/L}}$ since that is the concentration of the brine entering the system.