

Differential Equations - Exam #2 Practice Problems Answers

1. (a) $A(t)$ = Amount of salt in pounds at t minutes

$$\text{IVP: } \frac{dA}{dt} = 3 - \frac{3A}{200}, \quad A(0) = 100$$

- (b) The equilibrium solution of the DE is $A \equiv 200$. (Set the DE equal to 0 and solve for A .)

Here are a couple of ways to interpret the equilibrium solution:

- $A = 200$ is the limiting amount of salt in the tank. (As time goes on, we can expect the amount of salt in the tank to approach 200 pounds.)
- If the tank began with 200 pounds of salt, it would always have 200 pounds of salt (since the concentration of the brine initially matches the concentration of the brine flowing into the tank).

(c) IVP: $\frac{dA}{dt} = 3 - \frac{2A}{200+t}, \quad A(0) = 100$

The DE is linear $\Rightarrow A(t) = (200 + t) + C(200 + t)^{-2}$

Using the initial condition: $A(t) = (200 + t) - 4,000,000(200 + t)^{-2}$

The tank overflows at $t = 300$ minutes: $A(300) = 484$ pounds

$$\Rightarrow \text{Concentration} = \frac{484}{500} = 0.968 \text{ lb/gal}$$

2. $T(t)$ = internal temperature of the car in $^{\circ}\text{F}$ at t hours after turning off the car

Newton's Law of Cooling: $\frac{dT}{dt} = \frac{4}{3}[100 - T(t)]$ (Linear or separable)

$$\Rightarrow T(t) = 100 - 30e^{-4t/3}$$

Solving $T(t) = 95 \Rightarrow t = -\frac{3}{4} \ln\left(\frac{1}{6}\right) \approx 1.3438$ hours

3. (a) $v(t)$ = velocity of the object in m/s at t seconds

$$\text{IVP: } 100 \frac{dv}{dt} = 100(9.81) - 10v, \quad v(0) = 0$$

- (b) The equilibrium solution is $v \equiv \frac{100(9.81)}{10} = 98.1$. This is the terminal velocity of the object.

(c) Solving the IVP: $v(t) = 98.1 - 98.1e^{-t/10}$

Integrating to get the position function: $x(t) = 98.1t + 981e^{-t/10} - 981$ (Note: $x(0) = 0$)

4. Solution of homogeneous equation: $y_h = c_1 e^{-t/4} \cos(\frac{3}{4}t) + c_2 e^{-t/4} \sin(\frac{3}{4}t)$ (since $r = -\frac{1}{4} \pm \frac{3}{4}i$)

Particular solution: $y_p = \frac{8}{9}e^{-t}$ (Has form Ae^{-t} , Method of Undetermined Coeff. $\rightarrow A = \frac{8}{9}$)

General Solution: $y = y_h + y_p = c_1 e^{-t/4} \cos(\frac{3}{4}t) + c_2 e^{-t/4} \sin(\frac{3}{4}t) + \frac{8}{9}e^{-t}$

Using initial conditions: $y = -\frac{8}{9}e^{-t/4} \cos(\frac{3}{4}t) + \frac{20}{9}e^{t/4} \sin(\frac{3}{4}t) + \frac{8}{9}e^{-t}$

5. The auxiliary equation has solutions $r_1 = -1$ and $r_2 = -\frac{1}{2}$.

Looking at each term of the nonhomogeneity:

$$y_p = A_2 t^2 + A_1 t + A_0 + B_1 \cos t + B_2 \sin t + t(C_1 t + C_2)e^{-t}$$

6. The auxiliary equation has a double root $r = 3$.

Solution to the homogeneous equation: $y_h = c_1 e^{3t} + c_2 t e^{3t}$

Particular solution: $y_p = \frac{2}{3}t^2 + \frac{8}{9}t + \frac{4}{9} - 6t^2 e^{3t}$ (Has form $A_2 t^2 + A_1 t + A_0 + B t^2 e^{3t}$)

General Solution: $y = c_1 e^{3t} + c_2 t e^{3t} + \frac{2}{3}t^2 + \frac{8}{9}t + \frac{4}{9} - 6t^2 e^{3t}$

7. Differential equation modeling the system: $y'' + 4y' + 5y = 4 \cos(5t)$

Solution to the homogeneous equation: $y_h = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

Particular solution: $y_p = -\frac{1}{10} \cos(5t) + \frac{1}{10} \sin(5t)$

General Solution: $y = -\frac{1}{10} \cos(5t) + \frac{1}{10} \sin(5t) + c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

Using initial conditions: $y = -\frac{1}{10} \cos(5t) + \frac{1}{10} \sin(5t) + 1.1e^{-2t} \cos t + 1.7e^{-2t} \sin t$

8. (a) Example equation: $y'' + y' + 100y = 0$

(Any example of a linear second-order DE with constant coefficients for which the auxiliary equation has complex solutions/roots ($b^2 < 4mk$) will work).

(b) Example equation: $y'' - 2y' - 8y = 0$

(Any example of a linear second-order DE with constant coefficients for which the auxiliary equation has roots $r = 4$ and $r = -2$ will work).

(c) Example equation: $y'' + 7y' = 5te^{-7t}$

(Any example of a linear second-order DE with constant coefficients for which the auxiliary equation has a simple root of $r = -7$ and a nonhomogeneity involving a linear term multiplied by e^{-7t} will work).

(d) Example equation: $y'' + 5y' + y = 0$

(Any example of a linear second-order DE with constant coefficients for which the auxiliary equation has real-solutions ($b^2 > 4mk$) will work).