Math 207
Exam 1 Answers

1. Yes Using implicit differentiation: $1 + \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx} \quad \Rightarrow \quad 1 + y^2 + (1+y^2)\frac{dy}{dx} = \frac{dy}{dx}$

$$\Rightarrow \quad 1 + y^2 + y^2\frac{dy}{dx} = 0$$

2. (a) Since we have $\frac{dy}{dx} = f(x,y)$ with $f(x,y) = 10x - 2 - \frac{3y}{x}$ with f and $\frac{\partial f}{\partial y}$ continuous when $x = 1$ and $y = 4$, we can conclude that there is a unique solution to the IVP (with all those lovely domain requirements).

   (b) This equation is linear. Using the integrating factor $\mu(x) = x^3$ and the initial condition, we have the solution $y = 2x^2 - \frac{1}{2}x + \frac{5}{2}x^{-3}$.

3. (a) This equation is separable. The solutions are $y = -\ln(-\frac{1}{2}e^{x^2} + C)$.

   (b) This equation can be solved using special integrating factors.

   Since $\frac{\partial N/\partial x - \partial M/\partial y}{M} = -\frac{1}{y}$, $\mu(y) = \frac{1}{y}$ is the integrating factor that will make this equation exact.

   Solutions: $\sin^2 x + 2xy^2 = C \quad \text{or} \quad -\frac{1}{2}\cos(2x) + 2xy^2 = C$

   Don't forget to check $y \equiv 0$ because it may have been lost by multiplying by the integrating factor $\mu(y) = \frac{1}{y}$. Upon looking at the **original** equation, we can see that $y \equiv 0$ is a solution.

4. To show that $\frac{dy}{dx} = f(x,y)$ with $f(x,y) = \frac{x^3+3y^3}{2xy^2}$ is homogeneous, you can either test by showing that $f(tx,ty) = f(x,y)$ or by rewriting $f(x,y)$ as a function of $\frac{y}{x}$.

   Substituting $v = \frac{y}{x}$ or $y = vx$ $\left(\frac{dy}{dx} = x\frac{dv}{dx} + v\right)$:

$$x\frac{dv}{dx} + v = \frac{1+3v^3}{2v^2}$$

   This equation is separable with solutions $\frac{2}{3}\ln|1 + (\frac{y}{x})^3| = \ln|x| + C$

5. Let $B(t) =$ Account balance in dollars at time t, $\quad t =$ time in years

$$\frac{dB}{dt} = 0.05B - 10,000, \quad \text{with} \quad B(0) = 500,000$$

6. (a) The equilibrium solutions are $p \equiv 0, \quad p \equiv 1, \quad p \equiv 2.$

(b) Note that $\frac{dp}{dt} < 0$ when $p > 2$. (It may help to roughly sketch a direction field.) Since $p \equiv 2$ is an equilibrium solution, we must have that $\lim_{t\to\infty} p(t) = \boxed{2.}$*

*Note: Implicit in this is argument is the fact that $\frac{dp}{dt} = p(1-p)(p-2)$ satisfies the conditions of the existence/uniqueness theorem for any value of p and t, which means that given any point, only one solution can pass through that point. This helps us assert that starting with our initial condition, (0,3), our solution decreases, but cannot cross the line $y = 2$, which is a solution. So, the solution through (0,3) must have $y = 2$ as an asymptote.