Sequences and Series: Convergence or Divergence

Sequences: $\{a_1, a_2, a_3, ..., a_n, ...\} = \{a_n\}_{n=1}^{\infty}$
Informal Definition: A sequence $\{a_n\}$ converges to the number L, and we write $L = \lim_{n\to\infty} a_n$ if the terms a_n can be made as close to L as desired by taking n sufficiently large. Otherwise, the sequence diverges.
($\lim_{n\to\infty} a_n = \infty$ means that for any real number M, we have $a_n > M$ for all sufficiently large n.)

- Thm: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$. (11.1)
- Squeeze Thm: If $a_n \leq b_n \leq c_n$ for $n \geq N$ for some number N and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$. (11.1)
- Thm: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. (11.1)
- Thm: Every bounded, monotonic (increasing or decreasing) sequence is convergent. (11.1)
- Two particular sequences:
  1. $\{r^n\}$: $\lim_{n\to\infty} r^n = 0$, if $-1 < r < 1$
     $\lim_{n\to\infty} r^n = 1$, if $r = 1$ Otherwise, divergent (11.1)
  2. $\left\{\frac{c_1 n^p + ...}{c_2 n^q + ...}\right\}$ ($c_1 n^p$ = term with highest power in the numerator, $c_2 n^q$ = term with highest power in the denominator):
     $\lim_{n\to\infty} a_n = 0$, if $q > p$
     $\lim_{n\to\infty} a_n = \frac{c_1}{c_2}$, if $p = q$
     $\lim_{n\to\infty} a_n = \pm\infty$, if $p > q$

Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$

Definition: Let $s_n = \sum_{i=1}^{n} = a_1 + a_2 + ...a_n$. If the sequence $\{s_n\}$ is convergent with $\lim_{n\to\infty} s_n = S$, then $\sum a_n$ is convergent with $\sum_{n=1}^{\infty} a_n = S$. Otherwise, divergent.

- Thm: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

  $\Rightarrow$ Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges. (11.2)

- The Integral Test: For f continuous, positive, and decreasing on $[1, \infty)$ with $a_n = f(n)$,
  (i) if $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  (ii) if $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent. (11.3)

- Direct Comparison Test: For series $\sum a_n$ and $\sum b_n$:
  (i) If $0 \le a_n \le b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
  (ii) If $0 \le b_n \le a_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ is divergent. (11.4)

- Limit Comparison Test: For series $\sum a_n$ and $\sum b_n$ with positive terms, if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then either both series converge or both diverge. (11.4)

- Alternating Series Test: If the alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ $(b_n > 0)$ satisfies
  (i) $b_n \ge b_{n+1}$ for all n and (ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent. (11.5)

- Thm: If a series $\sum a_n$ is absolutely convergent, then it is convergent.
  (Absolutely convergent: $\sum |a_n|$ converges) (11.6)

- Ratio Test: For a series $\sum a_n$:
  (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series is absolutely convergent $\Rightarrow$ convergent.
  (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then the series is divergent.
  (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive. (11.6)

- Root Test: For a series $\sum a_n$:
  (i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series is absolutely convergent $\Rightarrow$ convergent.
  (ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then the series is divergent.
  (iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Ratio Test is inconclusive. (11.6)

- Two particular series:

  1. Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ...$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise. (11.2)
  2. P-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise. (11.3)