Math 126 Extra Credit Assignment Solutions

1. Does $\sum_{n=0}^{\infty}\frac{4\cdot5^n-5\cdot4^n}{6^n}$ converge? If so, what number does it converge to?

   Note that $\frac{4\cdot5^n-5\cdot4^n}{6^n}=\frac{4\cdot5^n}{6^n}-\frac{5\cdot4^n}{6^n}=4\left(\frac{5}{6}\right)^n-5\left(\frac{4}{6}\right)^n=4\left(\frac{5}{6}\right)^n-5\left(\frac{2}{3}\right)^n$

   Since $\left|\frac{5}{6}\right|<1$ and $\left|\frac{2}{3}\right|<1$, we know that

   $$\sum_{n=0}^{\infty}4\left(\frac{5}{6}\right)^n=\frac{4}{1-\frac{5}{6}}=24 \text{ (Geometric series with } a=4,\ r=\tfrac{5}{6})$$
   $$\sum_{n=0}^{\infty}5\left(\frac{2}{3}\right)^n=\frac{5}{1-\frac{2}{3}}=15 \text{ (Geometric series with } a=5,\ r=\tfrac{2}{3})$$

   So $\sum_{n=0}^{\infty}\frac{4\cdot5^n-5\cdot4^n}{6^n}=\sum_{n=0}^{\infty}4\left(\frac{5}{6}\right)^n-\sum_{n=0}^{\infty}5\left(\frac{2}{3}\right)^n=24-15=9.$

2. Determine whether the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^9-n^3+1}}$ is convergent or divergent and justify your answer.

   Since the series is similar to a p-series with $p=3$, comparison tests are a good choice.

   Limit Comparison Test with $a_n=\frac{1}{n^3}$:

   $$\lim_{n\to\infty}\left|\frac{\frac{1}{n^3}}{\frac{1}{\sqrt[3]{n^9-n^3+1}}}\right|=\lim_{n\to\infty}\frac{\sqrt[3]{n^9-n^3+1}}{n^3}=\lim_{n\to\infty}\sqrt[3]{\frac{n^9-n^3+1}{n^9}}=\lim_{n\to\infty}\sqrt[3]{1-\frac{1}{n^6}+\frac{1}{n^9}}=1$$

   Since $\lim_{n\to\infty}\frac{1}{n^3}$ converges, and we have a finite nonzero limit from our comparison test, the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^9-n^3+1}}$ converges.

3. For what values of x does the series $\sum_{n=0}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n}}$ converge?

   Ratio Test:

   $$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1}}}{\frac{2^n(x-3)^n}{\sqrt{n}}}\right|=\lim_{n\to\infty}\left|\frac{2(x-3)\sqrt{n}}{\sqrt{n+1}}\right|=2|x-3| \text{ since } \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}=0$$

   The ratio test states convergence of the original series when $2|x-3|<1 \Rightarrow \frac{5}{2}<x<\frac{7}{2}$.
   Checking the endpoints:
   For $x=\frac{5}{2}$, we have $\sum_{n=0}^{\infty}\frac{2^n(-\frac{1}{2})^n}{\sqrt{n}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test.

   For $x=\frac{7}{2}$, we have $\sum_{n=0}^{\infty}\frac{2^n(\frac{1}{2})^n}{\sqrt{n}}=\sum_{n=0}^{\infty}\frac{1^n}{\sqrt{n}}=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$ which diverges by the p-series test.

So the original series converges for x in the interval $[\frac{5}{2}, \frac{7}{2})$.

4. Find the Maclaurin series for the function $f(x) = x^5 - 5x^4 + 27x^2 - 3x + 17$. What can you say (in general) about a Maclaurin series for a polynomial?

   After computing the derivatives of f at $x = 0$ and using the equation for the Maclaurin series, you should find that the Maclaurin series for f is the polynomial f. This is true in general for polynomials. The Maclaurin series of a polynomial will always be the polynomial itself since it is already a series centered at 0 with only a finite number of nonzero terms.

5. For a vector $\vec{a} =< 6, 8 >$, find a vector $\vec{b}$ so that the vector projection of $\vec{b}$ onto $\vec{a}$ has length 3.

   If the vector projection of $\vec{b}$ onto $\vec{a}$ has length 3, then the scalar projection of $\vec{b}$ onto $\vec{a}$ will be either 3 or -3.

   Let $\vec{b} =< b_1, b_2 >$. Then the scalar projection of $\vec{b}$ onto $\vec{a}$ is $\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{6b_1+8b_2}{10}$.

   We want $\vec{b}$ so that $\text{comp}_{\vec{a}}\vec{b} = \pm 3 \quad \Rightarrow \quad \frac{6b_1+8b_2}{10} = \pm 3 \quad \Rightarrow \quad 6b_1 + 8b_2 = \pm 30$

   Any values of b_1 and b_2 that satisfy the above equation will work as components for $\vec{b}$. For example, we could choose $\vec{b} =< 1, 3 >$.

6. Find an equation of the plane that contains the point $(1, 0, -2)$ and is perpendicular to the line $y = 2x$ in the xy-plane.

   To write the equation of a plane, we need a point and a normal vector.

   We know that the line $y = 2x$ in the xy-plane is perpendicular to the plane, so the vector $< 1, 2, 0 >$ (or any multiple of $< 1, 2, 0 >$ is normal to the plane.

   So the equation of the plane is $1(x - 1) + 2(y - 0) + 0(z + 2) = 0$ OR $x + 2y = 1$.

7. Consider the vector $\vec{c} = \vec{a} \times (\vec{a} \times \vec{b})$ for nonzero vectors $\vec{a}$ and $\vec{b}$.
   True or False:

   (a) $\vec{c} \perp \vec{a}$

   (b) $\vec{c} \perp \vec{b}$

Justify your answer. (If you believe (a) or (b) is false, provide a counterexample.)

The cross product of two nonzero vectors will yield a vector perpendicular to both of the initial vectors. So, $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ and the vector $\vec{a} \times (\vec{a} \times \vec{b})$ is perpendicular to both $\vec{a}$ and $\vec{a} \times \vec{b}$. Thus (a) is true.

We can easily prove that (b) is false by counterexample:
For example, let $\vec{a} =< 1, 0, 0 >$, $\vec{b} =< 0, 1, 0 >$. Then $\vec{a} \times \vec{b} =< 0, 0, 1 >$ and $\vec{c} = \vec{a} \times (\vec{a} \times \vec{b}) =< 0, -1, 0 >$. Since $\vec{c} \cdot \vec{b} =< 0, -1, 0 > \cdot < 0, 1, 0 >= -1 \neq 0$, $\vec{c}$ is not perpendicular to $\vec{b}$.

Here are some low tech 3D figures relating these vectors: