Math 126 Exam 4 Solutions

1. (25 pts.) The temperature T in a room is measured in °C at a given point (x, y) where x and y are in feet. Values of the function $T = f(x, y)$ are given in the following table.

	15	20	25
10	25°	23°	22°
20	26°	21°	16°
30	27°	17°	10°

(a) (10 pts.) Use the table to find a linear approximation of the function f when x is near 20 and y is near 20.

The linearization of f at $(20, 20)$ is

$$L(x, y) = f(20, 20) + f_x(20, 20)(x - 20) + f_y(20, 20)(y - 20).$$

To approximate values, note that $f_x(20, 20) = \lim_{h\to 0} \frac{f(20+h,20)-f(20,20)}{h}$.

Using values from the table: $f_x(20, 20) \approx \frac{f(30,20)-f(20,20)}{10} = \frac{17-21}{10} = -\frac{2}{5}$ °C/ft

AND $f_x(20, 20) \approx \frac{f(10,20)-f(20,20)}{-10} = \frac{23-21}{-10} = -\frac{1}{5}$ °C/ft

So, $f_x(20, 20) \approx \frac{1}{2}[-\frac{2}{5} - \frac{2}{5}] = -\frac{3}{10} = -.3$ °C/ft

Similarly, $f_y(20, 20) = -1$ °C/ft.

$$L(x, y) = 21 - .3(x - 20) - (y - 20) \text{ OR } L(x, y) = -.3x - y + 47$$

(b) (5 pts.) Estimate the value of $f(18, 22)$.

$f(18, 22) \approx L(18, 22) = -.3(18) - 22 + 47 = 19.6$ °C.

(c) (10 pts.) Suppose a heat-dependent bug is sitting at $(20, 20)$. What is the rate of change of the temperature if the bug moves toward the point $(24, 17)$? Include units in your answer.

The direction from the point $(20, 20)$ towards the point $(24, 17)$ is
$\vec{v} =< 24 - 20, 17 - 20 >=< 4, -3 >$.

The unit vector with the same direction as $\vec{v}$ is $\vec{u} = \frac{\vec{v}}{|\vec{v}|} =< \frac{4}{5}, -\frac{3}{5} >$.

The gradient vector at $(20, 20)$ is $\nabla f(20, 20) \approx < -.3, -1 >$ (from part (a)).

So, the approximate rate of change in the direction $\vec{v}$ is

$$D_{\vec{u}} f(20, 20) = < -.3, -1 > \cdot < \frac{4}{5}, -\frac{3}{5} > = .36\ ^{\circ}\text{C/ft}$$

2. (15 pts.) Let $w = x^2 z e^{2y}$ where $x,\ y,$ and z are functions of s and t. For the values $s = -4$ and $t = 3$, we have the following information:

$$x = 2 \qquad y = 0 \qquad z = 3$$

$$\frac{\partial x}{\partial s} = 2 \qquad \frac{\partial y}{\partial s} = 3 \qquad \frac{\partial z}{\partial s} = 4$$

$$\frac{\partial x}{\partial t} = -1 \qquad \frac{\partial y}{\partial t} = 0 \qquad \frac{\partial z}{\partial t} = 2$$

Find $\frac{\partial w}{\partial t}$ when $s = -4$ and $t = 3$.

By the Chain Rule: $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$

First Partial Derivatives:

$$\frac{\partial w}{\partial x} = 2xze^{2y}, \qquad \frac{\partial w}{\partial y} = 2x^2 z e^{2y}, \qquad \frac{\partial w}{\partial z} = x^2 e^{2y}$$

For $s = -4$ and $t = 3$ ($x = 2,\ y = 0,\ z = 3$):

$$\frac{\partial w}{\partial x} = 12, \qquad \frac{\partial w}{\partial y} = 24, \qquad \frac{\partial w}{\partial z} = 4$$

So for $s = -4$ and $t = 3$, $\quad \frac{\partial w}{\partial t} = 12(-1) + 24(0) + 4(2) = -4.$

3. (15 pts.) Find an equation of a tangent plane to the surface $z^2 - 4x^2 + y^2 = 2$ at the point $(2, 3, -3)$.

Solving for z: $z = \pm\sqrt{2 + 4x^2 - y^2}$ (The surface is not the graph of a function. The positive square root describes the top half of the surface and the negative square root describes the bottom half.)

To find the tangent plane at $(2, 3, -3)$, we will use $z = -\sqrt{2 + 4x^2 - y^2}$. ($(2, 3, -3)$ lies on the bottom half of the surface.)

First Partial Derivatives: $\quad f_x = \frac{-4x}{\sqrt{2+4x^2-y^2}}$ and $f_y = \frac{y}{\sqrt{2+4x^2-y^2}}$

So $f_x(2, 3) = -\frac{8}{3}$ and $f_y(2, 3) = \frac{3}{3}$ =1.

Tangent plane at $(2, 3, -3)$: $z - (-3) = -\frac{8}{3}(x-2) + (y-3)$ OR $z = -\frac{8}{3}x - \frac{2}{3}$

4. (30 pts.) Sam and Frodo are on a pleasant day hike on Mount Doom. The shape of the mountain is given by the equation $z = 5000 - \frac{1}{2}x^2 + 10x - \frac{1}{4}y^2 + 15y$, where x, y, and z are in meters. Sam and Frodo are currently standing at the coordinates $(-30, -40, 3250)$.

   (a) (10 pts.) What is the direction of steepest ascent? What is the maximum rate of ascent?

   First Partial Derivatives: $f_x = -x + 10$ $\qquad f_y = -\frac{1}{2}y + 15$

   Gradient vector at $(-30, -40)$: $\nabla f(-30, -40) =< 40, 35 >$

   The direction of steepest ascent is the same direction as the gradient, so the direction of steepest ascent is $< 40, 35 >$.

   $$\begin{aligned}\text{The maximum rate of ascent is } |\nabla f| &= \sqrt{40^2 + 35^2} \\ &= \sqrt{2825} \\ &= 5\sqrt{113} \\ &\approx 53.1507 \text{ vert. meters per horiz. meter}\end{aligned}$$

   (b) (10 pts.) Find a nonzero direction $\vec{v} =< a, b >$ for which the rate of ascent is zero from the point that Sam and Frodo are standing.

   We are looking for a direction $\vec{v}$ for which the directional derivative of f at $(-30, -40)$ in that direction is 0.
   Let $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$. We want to find $\vec{v}$ so that $0 = D_{\vec{u}}f(-30, 40) =< 40, 35 > \cdot \vec{u}$
   $$=< 40, 35 > \cdot \frac{\vec{v}}{|\vec{v}|}$$
   $$= \frac{1}{|\vec{v}|} < 40, 35 > \cdot < a, b >$$

   $$\Rightarrow \quad 0 =< 40, 35 > \cdot < a, b >= 40a + 35b$$

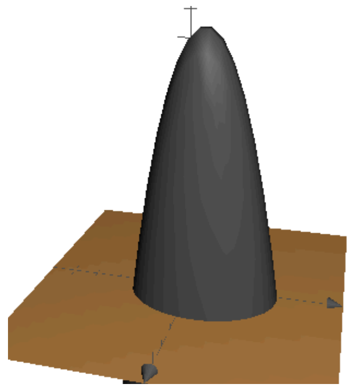
   So, we need to find values of a and b so that $40a + 35b = 0$. For example, we could choose $a = 7$ and $b = -8 \quad \Rightarrow \quad \vec{v} =< 7, -8 >$. In fact, $\vec{v}$ can be any vector that is a scalar multiple of $< 7, -8 >$.

   (c) (10 pts.) How tall is Mount Doom, i.e., what is the maximum z-value?

   We have that $f_x = -x + 10$ and $f_y = -\frac{1}{2}y + 15$. The first partials exist for all x and y, so the only critical points occur when $f_x = 0$ and $f_y = 0$.
   Since $f_x = 0$ only when $x = 10$ and $f_y = 0$ only when $y = 30$, the only critical point is $(10, 30)$.
   The 2nd derivative test shows that $(10, 30)$ is a local maximum:
   $$f_{xx} = -1, \; f_{yy} = -\frac{1}{2}, \; f_{xy} = 0 \quad \Rightarrow \quad D = (-1)(-\frac{1}{2}) - 0 > 0$$

In fact, we know that $(10, 30)$ is an absolute maximum since the graph of the surface is a downward-facing elliptic paraboloid.

So, the height of Mount Doom is $f(10, 30) = 5275$ meters.



Graph of Mount Doom with the plane $z = 0$

5. (15 pts.) Find the local maximum and minimum values and saddle points of the function $f(x, y) = 2x^3 + 5x^2 + xy + \frac{1}{2}y^2$.

First Partial Derivatives: $f_x = 6x^2 + 10x + y$ $\quad f_y = x + y$
The first partials exist for all x and y, so the critical points occur when $f_x = f_y = 0$.

$$\begin{aligned} f_x = 0 &\quad\Rightarrow\quad y = -6x^2 - 10x \\ f_y = 0 &\quad\Rightarrow\quad y = -x \end{aligned}$$

Intersecting the two curves: $-6x^2 - 10x = -x \quad \Rightarrow \quad 6x^2 + 9x = 0$

$$3x(2x + 3) = 0$$

$$x = 0 \text{ or } x = -\tfrac{3}{2}$$

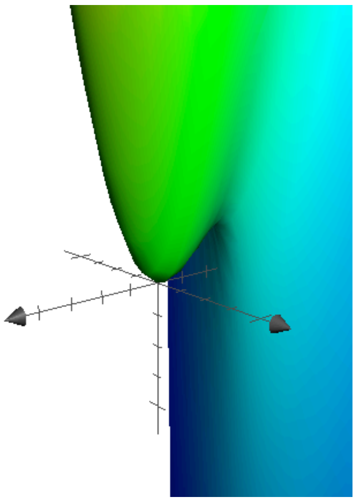
If $x = 0$, then $y = 0$ and if $x = -\frac{3}{2}$, then $y = \frac{3}{2}$. So, the critical points of f are $(0, 0)$ and $(-\frac{3}{2}, \frac{3}{2})$.

Second Partial Derivatives: $f_{xx} = 12x + 10$ $\quad f_{yy} = 1$ $\quad f_{xy} = 1$

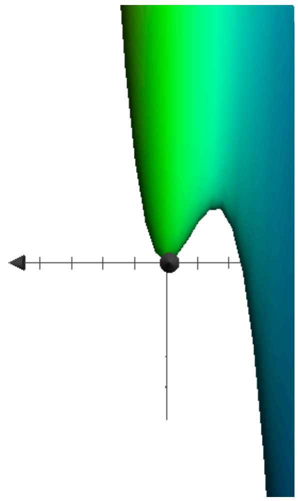
Second Derivative Test:
$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 10(1) - 1 > 0 \quad \Rightarrow \quad (0, 0)$ is a local minimum

$D(-\frac{3}{2}, \frac{3}{2}) = (-8)(1) - 1 < 0 \quad \Rightarrow \quad (-\frac{3}{2}, \frac{3}{2})$ is a saddle point.

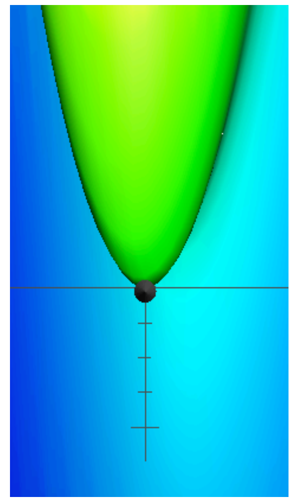
Graphs of f:



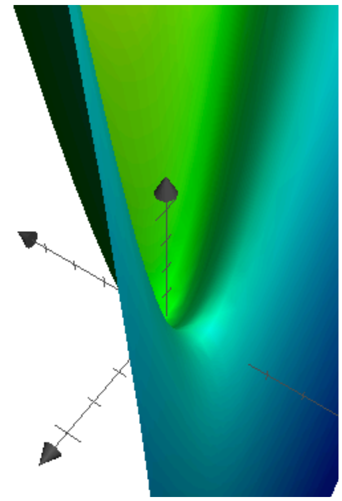
Standard view with the x and y-axis



View looking down the y-axis



View looking down the x-axis



View from above the z-axis