Math 126 Exam 3 Solutions

1. (20 pts.) $\vec{r}(t) = < e^t - 4,\ 2\sqrt{t} >$

   (a) (6 pts.) Find the velocity and acceleration vectors for the given position vector $\vec{r}$ at the point $(e-4, 2)$.

   The vector function $\vec{r}$ describes the point $(e-4, 2)$ when $t = 1$.

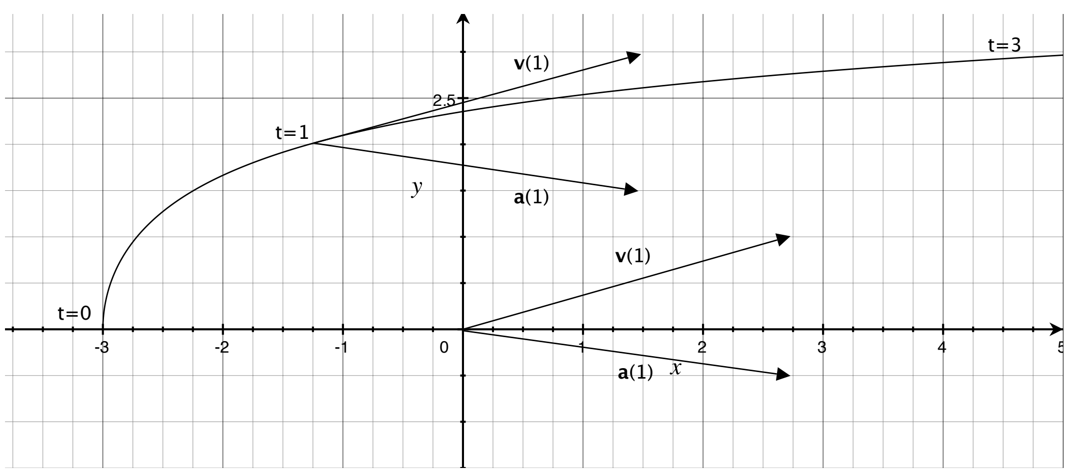
   $\vec{v}(t) = \vec{r}\,'(t) = < e^t, \frac{1}{\sqrt{t}} > \quad \Rightarrow \quad \vec{v}(1) = < e, 1 >$

   $\vec{a}(t) = \vec{v}\,'(t) = < e^t, -\frac{1}{2t^{3/2}} > \quad \Rightarrow \quad \vec{a}(1) = < e, -\frac{1}{2} >$

   (b) (14 pts.) Sketch the following on the axis below:

   i. The curve described by the vector function $\vec{r}$ for $t \geq 0$. Indicate with an arrow the direction in which the curve is traced as t increases.

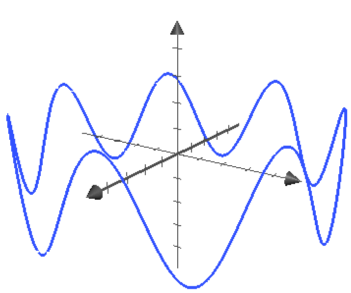
   ii. The velocity and acceleration vectors for $t = 1$.



   The velocity and acceleration vectors for $t = 1$ are graphed here originating from the origin and the point described by the position vector $\vec{r}(1)$.

2. (30 pts.) Sue has decided to go on a carnival ride called the Vominator. Placing coordinates so that the center of the ride is the origin, Sue's position on the ride is given by
$\vec{r}(t) = < 6\cos(t),\ 6\sin(t),\ 2\sin(7t) >$ where t is in seconds and the coordinates are in meters. The path of the ride is shown below.

   (a) (10 pts.) What is the **speed** at which Sue is moving at $\frac{\pi}{2}$ seconds?



To find the speed at $\frac{\pi}{2}$ seconds, we must find the velocity vector at $\frac{\pi}{2}$ seconds.

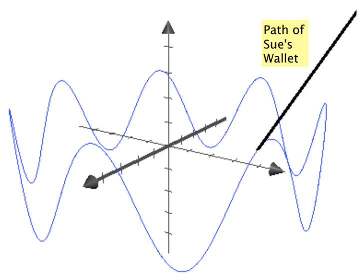
$\vec{r}\,'(t) =< -6\sin(t),\ 6\cos(t),\ 14\cos(7t) >$
So, $\vec{r}\,'(\frac{\pi}{2}) =< -6, 0, 0 >$ and $|\vec{r}\,'(\frac{\pi}{2})| = \sqrt{(-6)^2 + 0 + 0} = 6$ m/sec.

(b) (10 pts.) Set up and simplify, but do not evaluate, the integral needed to compute the total distance travelled by Sue from 0 seconds to 5 seconds.

For any given time t,
$$|\vec{r}\,'(t)| = \sqrt{(-6\sin(t))^2 + (6\cos(t))^2 + (14\cos(7t))^2} = \sqrt{36\sin^2(t) + 36\cos^2(t) + 196\cos^2(7t)} = \sqrt{36 + 196\cos(7t)}$$

So, the distance travelled from $t = 0$ to $t = 5$ seconds is given by the arclength formula:
Distance Travelled $= \int_0^5 |\vec{r}\,'(t)|\, dt = \int_0^5 \sqrt{36 + 196\cos(7t)}\, dt$

(c) (10 pts.) At $\frac{\pi}{3}$ seconds, Sue's wallet slips out of her pocket and assuming there is no air resistance or gravity, it travels on a trajectory tangent to the position curve (See figure). Where is the wallet located 1 second after it slips out of her pocket?

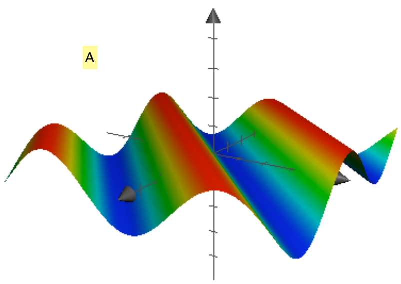


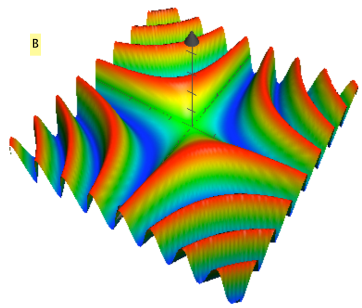
The wallet falls out at $\frac{\pi}{3}$ seconds, which means that it falls out when Sue is at the point described by $\vec{r}\,'(\frac{\pi}{3}) =< 6(\frac{1}{2}), 6(\frac{\sqrt{3}}{2}), 2(\frac{\sqrt{3}}{2}) >=< 3, 3\sqrt{3}, \sqrt{3} >$.
The velocity vector at the time $\frac{\pi}{3}$ seconds is
$\vec{r}\,'(\frac{\pi}{3}) =< -6(\frac{\sqrt{3}}{2}), 6(\frac{1}{2}), 14(\frac{1}{2}) >=< -3\sqrt{3}, 3, 7 >$. This velocity vector tells us how fast the wallet is moving in the direction of x, y, and z. The wallet is moving at $-3\sqrt{3}$ m/sec in the x-direction, 3 m/sec in the y-direction, and 7 m/sec in the z-direction. So, 1 second after the wallet falls out of her pocket, the wallet has moved $-3\sqrt{3}$ meters in the x-direction, 3 meters in the y-direction, and 7 meters in the z-direction or it has moved along the vector $< -3\sqrt{3}, 3, 7 >$.

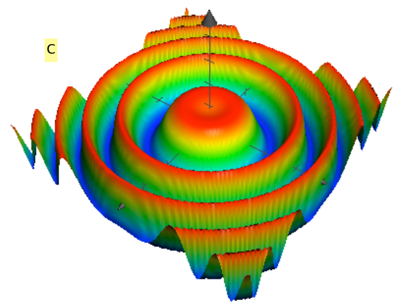
So, the position of the wallet is $(3 - 3\sqrt{3}, 3\sqrt{3} + 3, \sqrt{3} + 7)$ one second after it falls out of Sue's pocket. (Note the point above is also described by the vector sum of $\vec{r}(\frac{\pi}{3})$ and $\vec{r}\,'(\frac{\pi}{3})$.)

3. (10 pts.) Match the following functions to the graphs below.

- $\sin(x^2 + y^2 + 1)$
- $\sin(xy)$
- $\sin(x - y)$





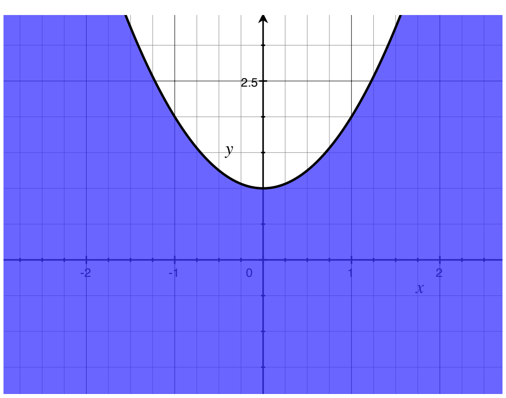


Looking at level curves for each function:

- The level curves for $\sin(x^2+y^2+1)$ will be circles centered at (0,0). So, the curve that matches this function must be **C**.
  (For example, if we take $k = 1$ and look at the level curve $\sin(x^2 + y^2 + 1) = k$, then we have $x^2 + y^2 + 1 = \frac{\pi}{2} + 2\pi n$ for any nonnegative integer n.)
- The level curves for $\sin(xy)$ will be hyperbolas. So, the curve that matches this function must be **B**.
  (For example, if we take $k = 1$ and look at the level curve $\sin(xy) = k$, then we have that $xy = \frac{\pi}{2} + 2\pi n$ for any integer n. So, along the curves $y = (\frac{\pi}{2} + 2\pi n)\frac{1}{x}$ we have the value $z = 1$.)
- The level curves for $\sin(x - y)$ will be lines with slope 1. So the curves that matches this function must be **A**.
  (For example, if we take $k = 1$ and look at the level curve $\sin(x - y) = k$, then we have $x - y = \frac{\pi}{2} + 2\pi n$ for any integer n or $y = x + \frac{\pi}{2} + 2\pi n$.)

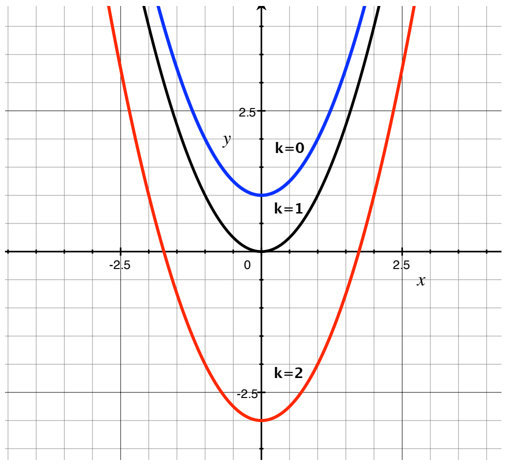
4. (20 pts.) $f(x, y) = \sqrt{1 + x^2 - y}$

(a) (10 pts.) Sketch the domain of the function in the axis below.



Domain: $\{(x, y) | y \leq 1 + x^2\}$

(b) (10 pts.) Sketch level curves ($f(x,y)=k$) on the axis below for $k=0,1$, and 2.



$\underline{k=0}$: $y=1+x^2$ $\underline{k=1}$: $y=x^2$ $\underline{k=2}$: $y=x^2-3$.

5. (20 pts.) $f(x,y)=x^2\ln(y)+e^{xy}$

(a) (7 pts.) Compute the first partial derivatives at (0,1).

$f_x(x,y)=2x\ln(y)+ye^{xy} \quad \Rightarrow \quad f_x(0,1)=0+e^0=1$

$f_y(x,y)=\frac{x^2}{y}+xe^{xy} \quad \Rightarrow \quad f_y(0,1)=0+0=0$

(b) (7 pts.) Compute $f_{xxy}(x,y)$.

$f_{xx}(x,y)=2\ln(y)+y^2e^{xy}$

$f_{xxy}(x,y)=\frac{2}{y}+2ye^{xy}+xy^2e^{xy}$

(c) (6 pts.) Compute $\frac{\partial^{50} f}{\partial x^{50}}$ (50th partial derivative of f with respect to x).

Note that $\frac{\partial^{50}}{\partial x^{50}}[x^2\ln(y)]=0$. In fact, $\frac{\partial^{3}}{\partial x^{3}}[x^2\ln(y)]=0$.

Also note that $\frac{\partial^{n}}{\partial x^{n}}[e^{xy}]=y^ne^{xy}$ for positive integers n.

So, $\frac{\partial^{50} f}{\partial x^{50}}=y^{50}e^{xy}$.