Math 126
Exam 1 Solutions

1. (30 pts.) Are the following series convergent or divergent? Justify your answers.

   (a) (10 pts.) $\sum_{n=1}^{\infty}(-1)^n\cos(\frac{\pi}{n})$

   This series diverges by the test for divergence since as $n \to \infty$, $\frac{\pi}{n} \to 0$ $\Rightarrow \cos(\frac{\pi}{n}) \to 1$. So, $\lim_{n\to\infty}(-1)^n\cos(\frac{\pi}{n})$ does not exist.

   (b) (10 pts.) $\sum_{n=0}^{\infty}\frac{(-2)^{n+1}}{5^n}$

   $$\sum_{n=0}^{\infty}\frac{(-2)^{n+1}}{5^n}=\sum_{n=0}^{\infty}\frac{(-2)(-2)^n}{5^n}=\sum_{n=0}^{\infty}(-2)\left(\frac{-2}{5}\right)^n$$

   This is a geometric series with $r = \frac{-2}{5}$ so it is convergent.

   Note: You can also easily prove convergence with the ratio test.

   (c) (10 pts.) $\sum_{n=0}^{\infty}\frac{1}{2^n-n^2}$

   As $n \to \infty$, 2^n will dominate the a_n terms since 2^n grows much faster than n^2 so we can expect convergence.

   Here are a couple of ways to prove convergence:

   - Compare this series with $\sum_{n=0}^{\infty}\frac{1}{2^n}$.

     Limit Comparison Test: $\lim_{n\to\infty}\frac{1}{2^n-n^2}\cdot\frac{2^n}{1}=\lim_{n\to\infty}\frac{2^n}{2^n-n^2}=\lim_{n\to\infty}\frac{1}{1-\frac{n^2}{2^n}}$

     Since $\lim_{n\to\infty}\frac{n^2}{2^n}=0$ (Can verify using L'Hospitals), $\lim_{n\to\infty}\frac{1}{1-\frac{n^2}{2^n}}=1>0$.

     Since $\sum_{n=0}^{\infty}\frac{1}{2^n}$ converges, we have that $\sum_{n=0}^{\infty}\frac{1}{2^n-n^2}$ converges.

   - Ratio test: $\lim_{n\to\infty}\left|\frac{1}{2^{n+1}-(n+1)^2}\cdot\frac{2^n-n^2}{1}\right|=\lim_{n\to\infty}\left|\frac{2^n-n^2}{2^{n+1}-(n+1)^2}\right|$.

     Here is one way to evaluate this limit: $\lim_{n\to\infty}\frac{2^n-n^2}{2^{n+1}-(n+1)^2}=\lim_{n\to\infty}\frac{1-\frac{n^2}{2^n}}{2-\frac{(n+1)^2}{2^n}}=\frac{1}{2}$

     since $\lim_{n\to\infty}\frac{n^2}{2^n}=\lim_{n\to\infty}\frac{(n+1)^2}{2^n}=0$.

Since $\frac{1}{2} < 1$, the ratio test tells us that we have absolute convergence $\Rightarrow$ convergence.

2. (20 pts.) Consider the sequence $\{a_n\}$ with $a_n = \dfrac{(3n)^n}{n^{3n}}$.

   (a) (10 pts.) Determine and prove whether the sequence $\{a_n\}$ converges or diverges. If it converges, find the limit.

   Note: You can simplify the terms a_n to $a_n = \dfrac{3^n n^n}{n^{3n}} = \dfrac{3^n}{n^{2n}} = \left(\dfrac{3}{n^2}\right)^n$.
   Here are a couple of ways to evaluate the limit:

   - Note that $\dfrac{3^n}{n^{2n}} = \dfrac{1}{\frac{n^{2n}}{3^n}} = \dfrac{1}{(\frac{n^2}{3})^n}$. Since $\left(\dfrac{n^2}{3}\right)^n \to \infty$ as $n \to \infty$, $\lim\limits_{n\to\infty} \dfrac{1}{(\frac{n^2}{3})^n} = 0$.

     Thus, $\lim\limits_{n\to\infty} \dfrac{(3n)^n}{n^{3n}} = 0$.

   - Note that $0 < \dfrac{3}{n^2} \leq \dfrac{3}{4}$ for $n \geq 2$. So $0 < \left(\dfrac{3}{n^2}\right)^n \leq \left(\dfrac{3}{4}\right)^n$.

     Since $\lim\limits_{n\to\infty} \left(\dfrac{3}{4}\right)^n = 0$, by the Squeeze Theorem for limits, we have that $\lim\limits_{n\to\infty} \left(\dfrac{3}{n^2}\right)^n = 0$.

     So $\lim\limits_{n\to\infty} \dfrac{(3n)^n}{n^{3n}} = 0$.

   - Another option to prove that $\lim\limits_{n\to\infty} \dfrac{(3n)^n}{n^{3n}} = 0$ is to do part (b) of this problem first. Since the series converges, we have that the terms a_n go to zero. So $\lim\limits_{n\to\infty} \dfrac{(3n)^n}{n^{3n}} = 0$.

   (b) (10 pts.) Determine and prove whether the series $\sum\limits_{n=1}^{\infty} a_n$ converges or diverges.

   Again, $a_n = \dfrac{3^n n^n}{n^{3n}} = \dfrac{3^n}{n^{2n}} = \left(\dfrac{3}{n^2}\right)^n$.

   Here are a few ways to prove convergence:

   - Comparison Test: $0 < \left(\dfrac{3}{n^2}\right)^n \leq \left(\dfrac{3}{4}\right)^n$ Since $\sum\limits_{n=1}^{\infty} \left(\dfrac{3}{4}\right)^n$ converges, $\sum\limits_{n=1}^{\infty} \dfrac{(3n)^n}{n^{3n}}$ converges.

   - Root Test: $\lim\limits_{n\to\infty} \sqrt[n]{\left|\dfrac{3^n}{n^{2n}}\right|} = \lim\limits_{n\to\infty} \dfrac{3}{n^2} = 0 \quad \Rightarrow$ The series converges absolutely.

   - Ratio Test: $\lim\limits_{n\to\infty} \left|\dfrac{3^{n+1}}{(n+1)^{2(n+1)}} \cdot \dfrac{n^2 n}{3^n}\right| = \lim\limits_{n\to\infty} \dfrac{3n^{2n}}{(n+1)^{2n+2}}$

     Note that $0 < \dfrac{3n^{2n}}{(n+1)^{2n+2}} < \dfrac{3n^{2n}}{n^{2n+2}} = \dfrac{3}{n^2} \to 0$.

So by the squeeze theorem for limits, $\lim_{n\to\infty} \frac{3n^{2n}}{(n+1)^{2n+2}} = 0 \quad \Rightarrow$ The series converges absolutely.

3. (15 pts.) Find the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^2}$.

   Using the Ratio Test: $\lim_{n\to\infty} \left| \frac{(4x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x-1)^n} \right| = \lim_{n\to\infty} \frac{|4x-1| n^2}{(n+1)^2} = |4x-1|$

   We get convergence when $|4x-1| < 1 \;\Rightarrow\; |x - \frac{1}{4}| < \frac{1}{4} \;\Rightarrow\; -\frac{1}{4} < x - \frac{1}{4} < \frac{1}{4} \;\Rightarrow\; 0 < x < \frac{1}{2}$.
   Checking the endpoints:
   $x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ Converges by A.S.T.
   $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{1^n}{n^2}$ Converges by p-series.

   So, the radius of convergence is $R = \frac{1}{4}$ and the interval of convergence is $0 \le x \le \frac{1}{2}$.

4. (15 pts.) If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = \frac{3n+2}{n}$, find a_n (for all n) and $\sum_{n=1}^{\infty} a_n$.

   Note: $a_1 = s_1 = 5$

   $$a_n = s_n - s_{n-1} = \frac{3n+2}{n} - \frac{3(n-1)+2}{n-1} = \frac{3n+2}{n} - \frac{3n-1}{n-1} = \frac{(3n+2)(n-1)-(3n-1)n}{n(n-1)}$$
   $$= \frac{-2}{n(n-1)} \text{ for } n \ge 2$$

   Since $\lim_{n\to\infty} s_n = 3$, $\sum_{n=1}^{\infty} a_n = 3$.

5. (20 pts.)

   (a) (10 pts.) Find a power series representation for the function $f(x) = \frac{4}{x-2}$ and determine the radius of convergence.

   $$f(x) = \frac{4}{x-2} = \frac{-2}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} (-2)\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} -\frac{x^n}{2^{n-1}} \text{ when } \left|\frac{x}{2}\right| < 1 \;\Rightarrow\; |x| < 2$$

   The radius of convergence is 2.

(b) (10 pts.) Find a power series representation for the function $g(x) = \frac{4}{(x-2)^2}$ and determine the radius of convergence.

Note that $f'(x) = \frac{-4}{x-2} = -g(x)$.

So $g(x) = -f'(x) = -\frac{d}{dx}[\sum_{n=0}^{\infty} -\frac{x^n}{2^{n-1}}] = \sum_{n=0}^{\infty} \frac{d}{dx}[\frac{x^n}{2^{n-1}}] = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n-1}}$ for $|x| < 2$ since the radius of convergence will be the same as with part (a).