Math 152
Final Exam Answers

Note: Many of these problems can be done in different ways. The following is just one way to do each problem.

1. Using washers: Each cross-section is a washer with an inner radius $r = 1$ and outer radius $R = e^{2x} + 1$. So, the area of each cross-section at x is given by $A(x) = \pi(e^{2x} + 1)^2 - \pi(1)^2$.

   $$\text{Volume of solid} = \int_0^2 \pi(e^{2x} + 1)^2 - \pi(1)^2\, dx = \pi \int_0^2 e^{4x} + 2e^{2x}\, dx$$
   $$= \pi[\ \frac{e^{4x}}{4} + e^{2x}\ ]_0^2$$
   $$= \boxed{\pi(\frac{e^8}{4} + e^4 - \frac{5}{4})}$$

2. Net Change in the Population over the next 504 months $= \int_0^{504} \frac{100}{1 + \sqrt[3]{t+8}}\, dt$

   $$= \int_2^8 \frac{300u^2}{1+u}\, du$$

   (Using the rationalizing substitution $u = \sqrt[3]{t+8} \quad \Rightarrow \quad 3u^2\ du = dt$)

   Using polynomial long division, we have that $\int_2^8 \frac{300u^2}{1+u}\, du = \int_2^8 300u - 300 + \frac{300}{1+u}\, du$

   $$= 150u^2 - 300u + 300ln|1+u|\ |_2^8$$
   $$= 7200 + 300 \ln 3$$
   $$\approx \boxed{9530 \text{ fish}}$$

3. Since $\frac{dy}{dx} = 4e^{4x}$, the arc length is given by $L = \int_0^1 \sqrt{1 + (4e^{4x})^2}\, dx = \int_0^1 \sqrt{1 + 16e^{8x}}\, dx$

   Using Simpson's Rule with $f(x) = \sqrt{1 + 16e^{8x}}$ and $\Delta x = \frac{1}{4}$:
   $$L \approx \frac{1}{12}(f(0) + 4f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 4f(\tfrac{3}{4}) + f(1))$$
   $$= \frac{1}{12}(\sqrt{1+16} + 4\sqrt{1+16e^2} + 2\sqrt{1+16e^4} + 4\sqrt{1+16e^6} + \sqrt{1+16e^8})$$
   $$\approx \boxed{53.8945}$$

4. Separating the variables: $\frac{dy}{dx} = \frac{x}{y(x^2+1)} \quad \Rightarrow \quad y\ dy = \frac{x\ dx}{x^2+1}$

$$\Rightarrow \quad \int y\,dy = \int \frac{x}{x^2+1}\,dx \quad \Rightarrow \quad \frac{y^2}{2} = \frac{1}{2}\ln(x^2+1) + C$$

(Note: I'm using the substitution $u = x^2+1$ on the second integral and I am not including the absolute values in the natural log since x^2+1 is always positive.)

Solving for y: $\quad y = \pm\sqrt{\ln(x^2+1)+C^*} \qquad$ (where $C^* = 2C$)

Using the initial condition, we have that $3 = \sqrt{\ln(1)+C^*} \quad \Rightarrow \quad C^* = 9$ (Using the positive square root since y is positive)

So, the solution is $\boxed{y = \sqrt{\ln(x^2+1)+9}.}$

5. (a) $$\int \frac{t+3}{(t+2)(t+1)} = \int \frac{-1}{t+2} + \frac{2}{t+1}\,dt \qquad \text{(Using partial fraction decomposition)}$$
$$= \boxed{-\ln|t+2| + 2\ln|t+1| + C}$$

(b) Consider $$\lim_{t\to\infty}\int_1^t \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to\infty}\int_1^{1/t} -u\cdot e^u\,du \qquad \left(\text{Subst. } u = \frac{1}{x} \quad \Rightarrow \quad du = -\frac{1}{x^2}\,dx\right)$$
$$= \lim_{t\to\infty} -ue^u + e^u \,\Big|_1^{1/t} \qquad \text{(Using int. by parts)}$$
$$= \lim_{t\to\infty} -\frac{1}{t}e^{1/t} + e^{1/t} + e - e$$
$$= \boxed{1} \qquad \text{since } \frac{1}{t} \to 0 \text{ as } t \to \infty$$

6. Finding the Intersection: Solving $\sqrt{3}\cos\theta = \sin\theta \quad \Rightarrow \quad \tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$

For $0 \le \theta \le \frac{\pi}{3}$, we need to find the area inside $r = \sin\theta$ since it is the innermost curve for those θ-values. For $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$, we need to find the area inside $r = \sqrt{3}\cos\theta$ since it is the innermost curve for those θ-values.

So, the area inside both is given by
$$\int_0^{\pi/3} \frac{1}{2}(\sin\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(\sqrt{3}\cos\theta)^2\,d\theta = \int_0^{\pi/3} \frac{1}{4}(1-\cos 2\theta)\,d\theta + \int_{\pi/3}^{\pi/2} \frac{3}{4}(1+\cos 2\theta)\,d\theta$$
(Using the half-angle trig. identities)
$$= \frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/3} + \frac{3}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\pi/3}^{\pi/2}$$
$$= \boxed{\frac{5\pi}{24} - \frac{\sqrt{3}}{4}}$$

7. (a) $F_{ave} = \frac{1}{9}\int_1^{10} F(x)\,dx = \frac{1}{9}$(Net Area from $x = 1$ to $x = 10$)

$$= \frac{1}{9}(11 - 4) = \boxed{\frac{7}{9}}$$

(b) The average rate of change of $F(x)$ from $x = 1$ to $x = 10$ is given by

$$\frac{F(10)-F(1)}{10-1} = \frac{-2-2}{9} = \boxed{-\frac{4}{9}.}$$

(c) FTC part 1 tells us that $f(x) = F'(x)$. So, to sketch $f(x)$, we need to sketch the derivative graph of $F(x)$.

The graph of the derivative of $F(x)$ would look as follows: