Math 152
Exam 2 Answers

NOTE: Many of these integrals can be evaluated in different ways. These solutions demonstrate one possibility.

1. $$\int \frac{5}{x+\sqrt{x+6}}\,dx = \int \frac{10u}{u^2-6+u}\,du \qquad \text{(Using subst. } u=\sqrt{x+6},\quad u^2-6=x \quad 2u\,du=dx)$$
$$= \int \frac{4}{u-2}+\frac{6}{u+3}\,du \qquad \text{(Partial Fraction Decomp.)}$$
$$= \boxed{4\ln|\sqrt{x+6}-2|+6\ln|\sqrt{x+6}+3|+C}$$
$$\text{OR } \boxed{4\ln|\sqrt{x+6}-2|+6\ln(\sqrt{x+6}+3)+C} \quad \text{since } \sqrt{x+6}+3 \text{ is positive}$$

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2. Using trigonometric substitution: $t = 2\sec\theta \quad \Rightarrow \quad dt = 2\sec\theta\tan\theta\, d\theta$,
$$\text{Average Value} = \frac{1}{4-2}\int_2^4 \frac{\sqrt{t^2-4}}{t^4}\,dt = \frac{1}{2}\int_0^{\pi/3} \frac{\sqrt{(2\sec\theta)^2-4}}{(2\sec\theta)^4}\cdot 2\sec\theta\tan\theta\,d\theta$$
$$= \frac{1}{8}\int_0^{\pi/3} \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$
$$= \frac{1}{8}\int_0^{\pi/3} \sin^2\theta\cos\theta\,d\theta$$
$$= \frac{1}{8}\int_0^{\sqrt{3}/2} u^2\,du \qquad \text{(Using subst. } u=\sin\theta...)$$
$$= \frac{1}{24}u^3\Big|_0^{\sqrt{3}/2} = \boxed{\frac{\sqrt{3}}{64}}$$

Note: I changed the bounds of integration with each of my substitutions, but you can choose not to. If you do not change your bounds, you need to return to the variable t and then evaluate at $t = 2$ and $t = 4$.

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3. Two Methods:

- Using Shells (Integrating with respect to x):
Radius of a shell at x: $r = x$ Height: $h = 4\arctan x$

$\Rightarrow$ Surface area of a shell at x: $2\pi x \cdot 4\arctan x$

$$\text{Volume} = \int_0^1 8\pi x\arctan x\,dx = 8\pi\left[\frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x\right]_0^1$$
$$= 8\pi\left[\frac{\pi}{4}-\frac{1}{2}\right] = \boxed{2\pi^2-4\pi \text{ units}^3}$$

Note: To evaluate the integral, use integration by parts ($u = \arctan x \quad \Rightarrow \quad dv = x\ dx$, $du = \frac{1}{1+x^2}\ dx, \quad v = \frac{x^2}{2}$) and then use polynomial long division on the resulting integral.

- Using Washers (Integrating with respect to y):

  Outer Radius of each washer at y: $R = 1$ Inner Radius: $r = \tan(\frac{y}{4})$

$$\text{Volume} = \int_0^{\pi} \pi[1 - \tan^2(\frac{y}{4})]\, dy = \int_0^{\pi} \pi[2 - \sec^2(\frac{y}{4})]\, dy$$
$$= \pi[2y - 4\tan(\tfrac{y}{4})]_0^{\pi}$$
$$= \pi[2\pi - 4] = \boxed{2\pi^2 - 4\pi \text{ units}^3}$$

Note: To evaluate the integral, use the identity $\tan^2\theta = \sec^2\theta - 1$.

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4. Position Function
$$s(t) = \int (9te^{3t} + \frac{4}{\sqrt{t+1}})\, dt$$
$$= \int 9te^{3t}\, dt + \int \frac{4}{\sqrt{t+1}}\, dt$$
$$= 3te^{3t} - e^{3t} + 8\sqrt{t+1} + C$$

To evaluate the first integral, use integration by parts ($u = 9t \quad \Rightarrow \quad dv = e^{3t}, \quad du = 9\ dt$, $v = \frac{1}{3}e^{3t}$).

To evaluate the second integral, you can use substitution with $w = t + 1$....

To find **the** position function with initial position 0 meters:

$$0 = s(0) = 3(0)e^0 - e^0 + 8\sqrt{1} + C \qquad \Rightarrow \qquad C = -7$$

The position function is $\boxed{s(t) = 3te^{3t} + e^{3t} + 8\sqrt{t+1} - 7.}$