Math 125 Worksheet #7 Solutions

1. Evaluate the following integrals using methods from §7.2 & 7.3.

(a) $\displaystyle\int \frac{\tan x}{\sec^2 x}\,dx$

Here are just a couple of ways to evaluate this integral:

- $$\begin{aligned}\int \frac{\tan x}{\sec^2 x}\,dx = \int \tan x\,\cos^2 x\,dx &= \int \frac{\sin x}{\cos x}\cdot \cos^2 x\,dx\\ &= \int \sin x\,\cos x\,dx\\ &= \int u\,du \quad (\text{Using substitution } u = \sin x \\ &\qquad\qquad\qquad\qquad \rightarrow \quad du = \cos x\;dx)\\ &= \tfrac{1}{2}u^2 + C\\ &= \tfrac{1}{2}\sin^2 x + C\end{aligned}$$

- $$\begin{aligned}\int \frac{\tan x}{\sec^2 x}\,dx = \int \tan x\,\cos^2 x\,dx &= \int \frac{\sin x}{\cos x}\cdot \cos^2 x\,dx\\ &= \int \sin x\,\cos x\,dx\\ &= \int \tfrac{1}{2}\sin(2x)\,dx \quad (\text{Trig. identity})\\ &= -\tfrac{1}{4}\cos(2x) + C\end{aligned}$$

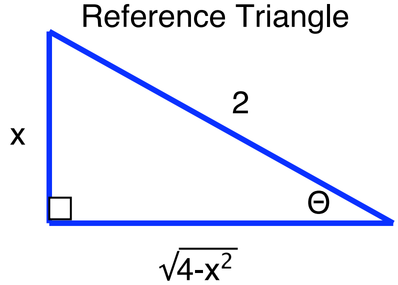
(b) $\displaystyle\int t^3\sqrt{4-t^2}\,dt$

Trigonometric Substitution: $t = 2\sin\theta \quad \rightarrow \quad dt = 2\cos\theta\;d\theta$

$$\begin{aligned}\int t^3\sqrt{4-t^2}\,dt &= \int 8\sin^3\theta\,\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta\\ &= \int 8\sin^3\theta\,\sqrt{4\cos^2\theta}\cdot 2\cos\theta\,d\theta\\ &= \int 8\sin^3\theta\cdot 2\cos\theta\cdot 2\cos\theta\,d\theta\\ &= 32\int \sin^3\theta\,\cos^2\theta\,d\theta\\ &= 32\int \sin\theta\,\sin^2\theta\,\cos^2\theta\,d\theta\\ &= 32\int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta\\ &= 32\int -(1-u^2)u^2\,du \quad (\text{Using substitution } u = \cos\theta\\ &\qquad\qquad\qquad\qquad \rightarrow \quad du = -\sin\theta\;d\theta)\\ &= 32\int u^4 - u^2\,du\\ &= 32[\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3] + C\end{aligned}$$

$$= 32[\tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta] + C$$

Reference Triangle



If $\sin\theta = \frac{x}{2}$, then $\cos\theta = \frac{\sqrt{4-x^2}}{2}$.

$$\Rightarrow \quad \int t^3\sqrt{4-t^2}\,dt = 32[\tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta] + C$$

$$= 32[\tfrac{1}{160}(\sqrt{4-x^2})^5 - \tfrac{1}{24}(\sqrt{4-x^2})^3] + C$$

$$= \tfrac{1}{5}(4-x^2)^{5/3} - \tfrac{4}{3}(4-x^2)^{3/2} + C$$

(c) $\displaystyle\int_{-3}^{0} \frac{x^3}{\sqrt{x^2+9}}\,dx$

Trigonometric Substitution: $\quad t = 3\tan\theta \quad \longrightarrow \quad dt = 3\sec^2\theta\ d\theta$

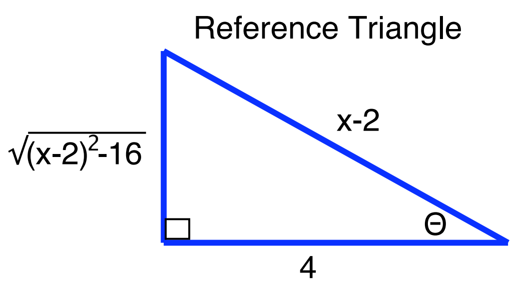
Note: $\quad t = 0 \quad \Rightarrow \quad 0 = 3\tan\theta \quad \Rightarrow \quad \theta = 0$
$\quad t = -3 \quad \Rightarrow \quad -3 = 3\tan\theta \quad \Rightarrow \quad \theta = -\frac{\pi}{4}$

$$\begin{aligned}
\int_{-3}^{0} \frac{x^3}{\sqrt{x^2+9}}\,dx &= \int_{-\pi/4}^{0} \frac{27\tan^3\theta}{\sqrt{9\tan^2\theta+9}} \cdot 3\sec^2\theta\,d\theta \\
&= \int_{-\pi/4}^{0} \frac{27\tan^3\theta}{\sqrt{9\sec^2\theta}} \cdot 3\sec^2\theta\,d\theta \\
&= \int_{-\pi/4}^{0} \frac{27\tan^3\theta}{3\sec\theta} \cdot 3\sec^2\theta\,d\theta \\
&= 27\int_{-\pi/4}^{0} \tan^3\theta\ \sec\theta\,d\theta \\
&= 27\int_{-\pi/4}^{0} \tan^2\theta\ \sec\theta\ \tan\theta\,d\theta \\
&= 27\int_{-\pi/4}^{0} (\sec^2 - 1)\ \sec\theta\ \tan\theta\,d\theta \\
&= 27\int_{\sqrt{2}}^{1} u^2 - 1\,du \qquad \text{(Using substitution } u = \sec\theta \\
&\qquad\qquad\qquad\qquad \longrightarrow \quad du = \sec\theta\ \tan\theta\ d\theta) \\
&= 27[\tfrac{1}{3}u^3 - u]_{\sqrt{2}}^{1} \\
&= 9u^3 - 27u\ |_{\sqrt{2}}^{1} \\
&= 9 - 27 - (9\cdot 2\sqrt{2} - 27\sqrt{2}) \\
&= -18 + 9\sqrt{2} \\
&\approx -5.2721
\end{aligned}$$

(d) $\int \frac{dx}{\sqrt{(x-2)^2-16}}$

Trigonometric Substitution: $x-2 = 4\sec\theta \quad \rightarrow \quad dx = 4\sec\theta\,\tan\theta\,d\theta$

$$\begin{aligned}
\int \frac{dx}{\sqrt{(x-2)^2-16}} &= \int \frac{4\sec\theta\,\tan\theta}{\sqrt{16\sec^2\theta-16}}\,d\theta \\
&= \int \frac{4\sec\theta\,\tan\theta}{\sqrt{16\tan^2\theta}}\,d\theta \\
&= \int \frac{4\sec\theta\,\tan\theta}{4\tan\theta}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta|+C
\end{aligned}$$



If $\sec\theta = \frac{x-2}{4}$, then $\tan\theta = \frac{\sqrt{(x-2)^2-16}}{4}$.

$$\begin{aligned}
\Rightarrow \quad \int \frac{dx}{\sqrt{(x-2)^2-16}} &= \ln|\sec\theta+\tan\theta|+C \\
&= \ln\left|\frac{x-2}{4} + \frac{\sqrt{(x-2)^2-16}}{4}\right|+C
\end{aligned}$$