# Math 125 Worksheet #6 Solutions

1. Evaluate the following integrals using integration by parts.

   (a) $\int_0^2 xe^{-x}\,dx$

   Integration by Parts: $u = x \quad \rightarrow \quad du = dx$
   $dv = e^{-x}\ dx \quad \rightarrow \quad v = -e^{-x}$

   $$\begin{aligned}\Rightarrow \int_0^2 xe^{-x}\,dx &= -xe^{-x}\ |_0^2 - \int_0^2 -e^{-x}\,dx \\ &= -xe^{-x}\ |_0^2 + \int_0^2 e^{-x}\,dx \\ &= -2e^{-2} + 0e^0 - e^{-x}\ |_0^2 \\ &= -2e^{-2} - e^{-2} + e^0 \\ &= -3e^{-2} + 1 \\ &\approx .59399\end{aligned}$$

   (b) $\int x^2(\ln\,x)^2\,dx$

   Integration by Parts: $u = (\ln\,x)^2 \quad \rightarrow \quad du = 2(\ln\,x)(\frac{1}{x})\ dx$
   $dv = x^2\ dx \quad \rightarrow \quad v = \frac{1}{3}x^3$

   $$\Rightarrow \int x^2(\ln\,x)^2\,dx = \tfrac{1}{3}x^3(\ln\,x)^2 - \tfrac{2}{3}\int x^2\ln\,x\,dx$$

   Integration by Parts (again): $u = \ln\,x \quad \rightarrow \quad du = \frac{1}{x}\ dx$
   $dv = x^2\ dx \quad \rightarrow \quad v = \frac{1}{3}x^3$

   $$\begin{aligned}\Rightarrow \int x^2(\ln\,x)^2\,dx &= \tfrac{1}{3}x^3(\ln\,x)^2 - \tfrac{2}{3}[\tfrac{1}{3}x^3\ln\,x - \tfrac{1}{3}\int x^2\,dx] \\ &= \tfrac{1}{3}x^3(\ln\,x)^2 - \tfrac{2}{9}x^3\ln\,x + \tfrac{2}{9}\int x^2\,dx \\ &= \tfrac{1}{3}x^3(\ln\,x)^2 - \tfrac{2}{9}x^3\ln\,x + \tfrac{2}{27}x^3 + C\end{aligned}$$

   (c) $\int \sec^2\sqrt{t}\,dt$ (Substitution and integration by parts will be needed.)

   Using substitution: $w = \sqrt{t} \quad \rightarrow \quad dw = \frac{dt}{2\sqrt{t}} \quad \rightarrow \quad 2\sqrt{t}\ dw = dt$ or $2w\ dw = dt$

   $$\Rightarrow \int \sec^2\sqrt{t}\,dt = 2\int w\ \sec^2 w\,dw$$

   Integration by Parts: $u = w \quad \rightarrow \quad du = dw$
   $dv = \sec^2 w\ \ dw \quad \rightarrow \quad v = \tan\,w$

   $$\Rightarrow \int \sec^2\sqrt{t}\,dt = 2[w\ \tan\,w - \int \tan\,w\,dw]$$

$$= 2w \tan w - 2\ln|\sec w| + C$$

$$= 2\sqrt{t} \tan\sqrt{t} - 2\ln|\sec\sqrt{t}| + C$$

2. Evaluate the following trigonometric integrals.

   (a) $\int \cos^3 x \sin^2 x \, dx$

   Since the integrand has odd powers of cosine:

$$\begin{aligned} \int \cos^3 x \sin^2 x \, dx &= \int \cos x \cos^2 x \sin^2 x \, dx \\ &= \int \cos x (1 - \sin^2 x) \sin^2 x \, dx \end{aligned}$$

   Using substitution: $u = \sin x \quad \rightarrow \quad du = \cos x \, dx$

$$\begin{aligned} \Rightarrow \quad \int \cos^3 x \sin^2 x \, dx &= \int (1 - u^2) u^2 \, du \\ &= \int u^2 - u^4 \, du \\ &= \tfrac{1}{3} u^3 - \tfrac{1}{5} u^5 + C \\ &= \tfrac{1}{3} \sin^3 x - \tfrac{1}{5} \sin^5 x + C \end{aligned}$$

   (b) $\int \sin^2\theta \cos^2\theta \, d\theta$

   There are a couple of ways to evaluate this integral.

   - Use the half-angle formulas: $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$, $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$$\begin{aligned} \int \sin^2\theta \cos^2\theta \, d\theta &= \int \frac{1 - \cos(2\theta)}{2} \cdot \frac{1 + \cos(2\theta)}{2} \, d\theta \\ &= \tfrac{1}{4} \int 1 - \cos^2(2\theta) \, d\theta \\ &= \tfrac{1}{4} \int 1 - \frac{1 + \cos(4\theta)}{2} \, d\theta \quad \text{(Half-angle formula once again)} \\ &= \tfrac{1}{8} \int 1 - \cos(4\theta) \, d\theta \\ &= \tfrac{1}{8} [\theta - \tfrac{1}{4} \sin(4\theta)] + C \\ &= \tfrac{1}{8} \theta - \tfrac{1}{32} \sin(4\theta) + C \end{aligned}$$

   - Use the double-angle formula: $\sin\theta \cos\theta = \frac{1}{2} \sin(2x)$

$$\begin{aligned} \int \sin^2\theta \cos^2\theta \, d\theta &= \int (\sin\theta \cos\theta)^2 \, d\theta \\ &= \int (\frac{1}{2} \sin(2\theta))^2 \, d\theta \end{aligned}$$

$$= \tfrac{1}{4}\int \sin^2(2\theta)\, d\theta$$

$$= \tfrac{1}{4}\int \frac{1-\cos(4\theta)}{2}\, d\theta \quad \text{(Half-angle formula)}$$

$$= \tfrac{1}{8}[\theta - \tfrac{1}{4}\sin(4\theta)] + C$$

$$= \tfrac{1}{8}\theta - \tfrac{1}{32}\sin(4\theta) + C$$