Math 125 Worksheet #2 Solutions

1. Find the most general form of a function f such that $f''(x) = 2e^x - 12x$.

   If $f''(x) = 2e^x - 12x$, then $f'(x) = 2e^x - \frac{12}{2}x^2 + C$
   $= 2e^x - 6x^2 + C$ for some arbitrary constant C.

   If $f'(x) = 2e^x - 6x^2 + C$, then $f(x) = 2e^x - \frac{6}{3}x^3 + Cx + D$
   $= 2e^x - 2x^3 + Cx + D$ for arbitrary constants C and D.

   Check: If $f(x) = 2e^x - 2x^3 + Cx + D$, then $f'(x) = 2e^x - 6x^2 + C$ and $f''(x) = 2e^x - 12x$. ✓

2. Find the area under the curve $f(x) = \frac{8}{x^3} + 4x$ between $x = 1$ and $x = 3$.

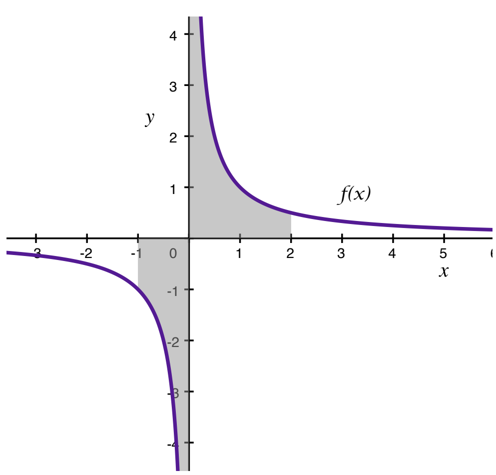
   The area under the curve is given by $\int_1^3 \frac{8}{x^3} + 4x\,dx = \int_1^3 8x^{-3} + 4x\,dx$

   $$= \frac{8}{-2}x^{-2} + \frac{4}{2}x^2\Big|_1^3$$
   $$= -\frac{4}{x^2} + 2x^2\Big|_1^3$$
   $$= -\frac{4}{3^2} + 2(3^2) - [-\frac{4}{1^2} + 2(1^2)]$$
   $$= -\frac{4}{9} + 20 = \frac{176}{9}$$

3. Consider $\int_{-1}^{2} \frac{1}{x}\,dx$. Can you evaluate this integral using the FTC? If so, what do you get? If not why? (See the graph of $f(x) = \frac{1}{x}$.) Does the definite integral $\int_{-1}^{2} \frac{1}{x}\,dx$ have a finite value?

   Note that if you tried to evaluate using the FTC, you would get the following:
   $$\int_{-1}^{2} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{2} = \ln|2| - \ln|-1| = \ln 2.$$

   However, we cannot use the FTC since f has an infinite discontinuity at $x = 0$. See the graph below.



   The shaded region is between the curve and the x-axis.

   Note that $\int_{-1}^{0} \frac{1}{x}\,dx$ and $\int_{0}^{2} \frac{1}{x}\,dx$ do not exist (the integrals are not finite). So, $\int_{-1}^{2} \frac{1}{x}\,dx$ does not exist. We will discuss definite integrals involving infinite discontinuities more later.

4. Find $\int_{-1}^{3} h(t)\,dt$ for $h(t) = \begin{cases} t+2 & \text{if } t \le 1 \\ 3t^2 & \text{if } t > 1 \end{cases}$.

Note that $h(t)$ is continuous at $t = 1$.

Also note that
$$\begin{aligned} \int_{-1}^{3} h(t)\,dt &= \int_{-1}^{1} h(t)\,dt + \int_{1}^{3} h(t)\,dt \\ &= \int_{-1}^{1} t+2\,dt + \int_{1}^{3} 3t^2\,dt \\ &= [\tfrac{1}{2}t^2 + 2t]_{-1}^{1} + t^3|_1^3 \\ &= \tfrac{1}{2}(1)^2 + 2(1) - [\tfrac{1}{2}(-1)^2 + 2(-1)] + 3^3 - 1^3 = 30 \end{aligned}$$

5. Let $g(x) = \int_0^x e^t - 5\sin\, t\,dt$ for $0 \le x \le \pi$.

(a) Find $g(\pi)$.

$$\begin{aligned} g(\pi) = \int_0^{\pi} e^t - 5\sin\, t\,dt &= e^t + 5\cos\, t|_0^{\pi} \\ &= e^{\pi} + 5\cos(\pi) - (e^0 + 5\cos(0)) \\ &= e^{\pi} - 11 \end{aligned}$$

(b) Find $g'(x)$.

Here are a couple of ways to find $g'(x)$:

- Given the first part of the FTC, we have that $g'(x)$ is equal to the integrand evaluated at x. $\Rightarrow$ $g'(x) = e^x - 5\sin\, x$.
- For a given x, $$\begin{aligned} g(x) = \int_0^x e^t - 5\sin\, t\,dt &= e^t + 5\cos\, t\,|_0^x \\ &= e^x + 5\cos\, x - (e^0 + 5\cos\, 0) \\ &= e^x + 5\cos\, x - 6 \end{aligned}$$

  So, $g'(x) = \frac{d}{dx}[e^x + 5\cos\, x - 6] = e^x - 5\sin\, x$.