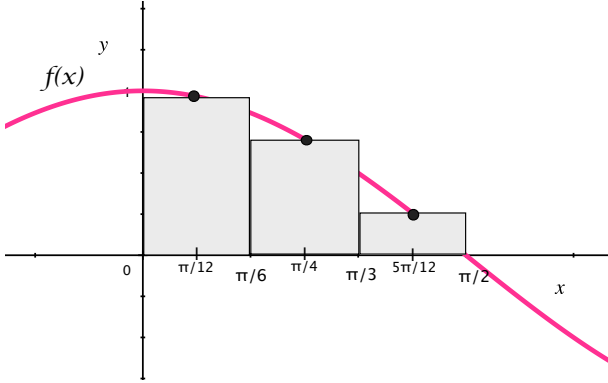


Math 125 Worksheet #1 Solutions

1. $f(x) = \cos(x)$

- (a) Estimate the area (to 4 decimal places) under the curve of f from $x = 0$ to $x = \frac{\pi}{2}$ using 3 approximating rectangles using midpoints as sample points.

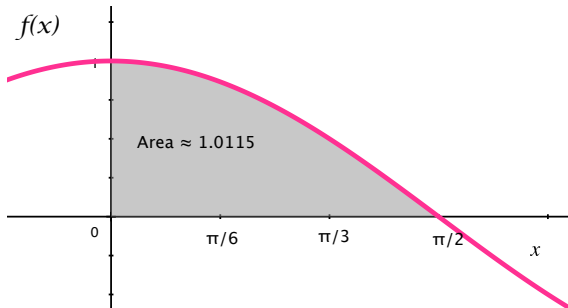


Using 3 rectangles, we have that

$$\Delta x = \frac{\pi}{6}.$$

Using the midpoints as sample points, we have that the sum of the areas of the approximating rectangles is given by the following:

$$\begin{aligned} \text{Sum} &= f\left(\frac{\pi}{12}\right) \cdot \frac{\pi}{6} + f\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{6} + f\left(\frac{5\pi}{12}\right) \cdot \frac{\pi}{6} = \frac{\pi}{6} [f\left(\frac{\pi}{12}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{5\pi}{12}\right)] \\ &= \frac{\pi}{6} \left[\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{5\pi}{12}\right) \right] \\ &\approx 1.0115 \end{aligned}$$

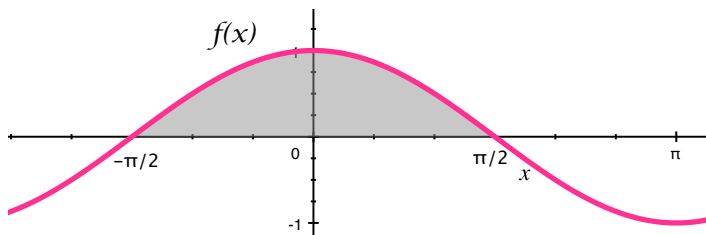


- (b) Approximate each of the following integrals given your approximation to part (a).

i. $\int_0^{\pi/2} \cos(x) dx$

By definition of the definite integral, we have that $\int_0^{\pi/2} \cos(x) dx = \text{area under the curve of } f \text{ between } x = 0 \text{ and } x = \frac{\pi}{2}$. So, $\int_0^{\pi/2} \cos(x) dx \approx 1.0115$.

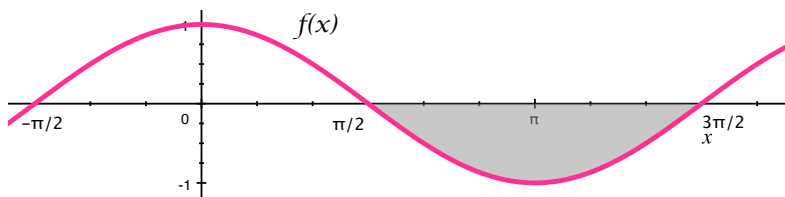
ii. $\int_{-\pi/2}^{\pi/2} \cos(x) dx$



The area of the shaded region in the diagram above is the value of the definite integral.

By symmetry, we have that $\int_{-\pi/2}^{\pi/2} \cos(x) dx \approx 2(1.0115) = 2.023$.

iii. $\int_{\pi/2}^{3\pi/2} \cos(x) dx$

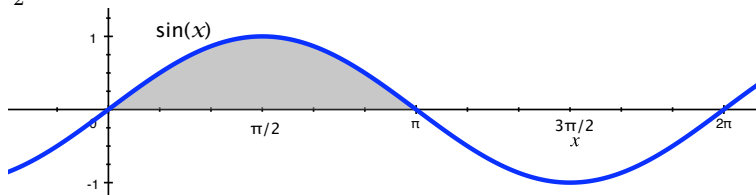


The area of the shaded region in the diagram above is the value of the definite integral.

By symmetry, we have that the shaded region has an approximate area of 2.023. Since the shaded area is below the x -axis, $\int_{\pi/2}^{3\pi/2} \cos(x) dx \approx -2.023$.

iv. $\int_0^{\pi} \sin(x) dx$

Note that the graph of $\sin(x)$ is the same as the graph of $\cos(x)$ shifted to the right by $\frac{\pi}{2}$.



The area of the shaded region in the diagram above is the value of the definite integral.

So, $\int_0^{\pi} \sin(x) dx = \int_{-\pi/2}^{\pi/2} \cos(x) dx \approx 2.023$.

2. Prove that for $b \geq a$, $\int_a^b x dx = \frac{1}{2}(b^2 - a^2)$ using the definition of area involving the limit of a sum. (You can verify this result graphically.)

By definition, $\int_a^b x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot \Delta x$ (Using right-end rectangles for $f(x) = x$.)

Note that $\Delta x = \frac{b-a}{n} \Rightarrow x_i = a + i\left(\frac{b-a}{n}\right)$.

So, we have that $\int_a^b x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + i\left(\frac{b-a}{n}\right)\right) \cdot \left(\frac{b-a}{n}\right)$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n (a + i(\frac{b-a}{n})) \cdot (\frac{b-a}{n}) && \text{(Evaluating the function } f.) \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n a(\frac{b-a}{n}) + i(\frac{b-a}{n})^2 \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n a(\frac{b-a}{n}) + \lim_{n \rightarrow \infty} \sum_{i=1}^n i(\frac{b-a}{n})^2 \\
&= \lim_{n \rightarrow \infty} a(\frac{b-a}{n}) \sum_{i=1}^n 1 + \lim_{n \rightarrow \infty} (\frac{b-a}{n})^2 \sum_{i=1}^n i \\
&= \lim_{n \rightarrow \infty} a(\frac{b-a}{n}) \cdot n + \lim_{n \rightarrow \infty} (\frac{b-a}{n})^2 \cdot \frac{n(n+1)}{2} && \text{(Using summation formulas.)} \\
&= \lim_{n \rightarrow \infty} a(b-a) + \lim_{n \rightarrow \infty} \frac{(b-a)^2}{n^2} \cdot \frac{n^2+n}{2} \\
&= a(b-a) + \lim_{n \rightarrow \infty} \frac{1}{2}(b-a)^2(1 + \frac{1}{n}) \\
&= ab - a^2 + \frac{1}{2}(b-a)^2 = \frac{1}{2}b^2 - \frac{1}{2}a^2 = \frac{1}{2}(b^2 - a^2)
\end{aligned}$$

3. Evaluate the following sums.

(a) $\sum_{j=2}^6 \tan(j\pi)$

Note that $\tan(j\pi) = 0$ for any integer j since $\sin(j\pi) = 0$ for any integer j .

So, $\sum_{j=2}^6 \tan(j\pi) = \sum_{j=2}^6 0 = 0 + 0 + 0 + 0 + 0 = 0$.

(b) $\sum_{i=1}^n (3i - 2)$ for some integer n

$$\begin{aligned}
\sum_{i=1}^n (3i - 2) &= 3 \sum_{i=1}^n i - \sum_{i=1}^n 2 = 3 \sum_{i=1}^n i - 2 \sum_{i=1}^n 1 \\
&= 3 \frac{n(n+1)}{2} - 2n && \text{(Summation formulas.)}
\end{aligned}$$

4. Find the most general form of a function f such that $f''(x) = 2e^x - 12x$.

If $f''(x) = 2e^x - 12x$, then $f'(x) = 2e^x - \frac{12}{2}x^2 + C$
 $= 2e^x - 6x^2 + C$ for some arbitrary constant C .

If $f'(x) = 2e^x - 6x^2 + C$, then $f(x) = 2e^x - \frac{6}{3}x^3 + Cx + D$
 $= 2e^x - 2x^3 + Cx + D$ for arbitrary constants C and D .

Check: If $f(x) = 2e^x - 2x^3 + Cx + D$, then $f'(x) = 2e^x - 6x^2 + C$ and $f''(x) = 2e^x - 12x$. ✓