Math 125 Worksheet #10 Solutions

1. Evaluate the following improper integrals.

   (a) $\displaystyle\int_1^\infty \frac{dx}{x \ln x}$

   This is improper for two reasons: There is a vertical asymptote at $x = 1$ and the upper bound is infinite. To evaluate, we must consider two improper integrals of the form $\displaystyle\int_1^a \frac{dx}{x \ln x}$ and $\displaystyle\int_a^\infty \frac{dx}{x \ln x}$ for some value a in the interval $(1, \infty)$. For simplicity, I will use $a = e$.

   - Consider $\displaystyle\lim_{t\to 1^+} \int_t^e \frac{dx}{x \ln x} = \lim_{t\to 1^+} \int_{\ln t}^1 \frac{du}{u}$ Substitution: $u = \ln x$, $du = \frac{1}{x}\,dx$

     $$\begin{aligned} &= \lim_{t\to 1^+} \ln u \,\Big|_{\ln t}^1 \\ &= \lim_{t\to 1^+} \ln 1 - \ln(\ln t) \\ &= \lim_{t\to 1^+} -\ln(\ln t) \\ &= \infty \quad \text{since } \ln t \to 0 \quad \Rightarrow \quad \ln(\ln t) \to -\infty \text{ as } t \to 1^+ \end{aligned}$$

   - Consider $\displaystyle\lim_{t\to \infty} \int_e^t \frac{dx}{x \ln x} = \lim_{t\to \infty} \int_1^{\ln t} \frac{du}{u}$ Substitution: $u = \ln x$, $du = \frac{1}{x}\,dx$

     $$\begin{aligned} &= \lim_{t\to \infty} \ln u \,\Big|_1^{\ln t} \\ &= \lim_{t\to \infty} \ln(\ln t) - \ln 1 \\ &= \lim_{t\to \infty} \ln(\ln t) \\ &= \infty \quad \text{since } \ln t \to \infty \quad \Rightarrow \quad \ln(\ln t) \to \infty \text{ as } t \to \infty \end{aligned}$$

   Note: It is unnecessary to compute both of the above improper integrals. Once it is known that one of the improper integrals is infinite, then we can determine that $\displaystyle\int_1^\infty \frac{dx}{x \ln x}$ is divergent.

   (b) $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$

   Note: The integrand has a vertical asymptote at $x = 0$.

   Consider $\displaystyle\lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \int_{\ln t}^0 u\,du$ Substitution: $u = \ln x$, $du = \frac{1}{x}\,dx$

   $$\begin{aligned} &= \lim_{t\to 0^+} \tfrac{1}{2}u^2 \,\Big|_{\ln t}^0 \\ &= \lim_{t\to 0^+} 0 - \tfrac{1}{2}(\ln t)^2 \\ &= -\infty \qquad \text{since } \ln t \to -\infty \text{ as } t \to 0^+ \end{aligned}$$

   Thus, $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$ diverges since it is infinite.

(c) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

To calculate this improper integral, we must compute two improper integrals of the form $\displaystyle\int_{-\infty}^{a} xe^{-x^2}\,dx$ and $\displaystyle\int_{a}^{\infty} xe^{-x^2}\,dx$ for some value a. For simplicity, I will use $a = 0$.

- Consider $\displaystyle\lim_{t\to-\infty}\int_{t}^{0} xe^{-x^2}\,dx = \lim_{t\to-\infty} -\tfrac{1}{2}\int_{-t^2}^{0} e^u\,du$ Substitution: $u = -x^2$, $du = -2x\,dx$

$$
\begin{aligned}
&= \lim_{t\to-\infty} -\tfrac{1}{2}e^u \,\Big|_{-t^2}^{0}\\
&= \lim_{t\to-\infty} -\tfrac{1}{2}e^0 + \tfrac{1}{2}e^{-t^2}\\
&= -\frac{1}{2} \quad \text{since } e^{-t^2} \to 0 \text{ as } t \to -\infty
\end{aligned}
$$

- Consider $\displaystyle\lim_{t\to\infty}\int_{0}^{t} xe^{-x^2}\,dx = \lim_{t\to-\infty} -\tfrac{1}{2}\int_{0}^{-t^2} e^u\,du$ Substitution: $u = -x^2$, $du = -2x\,dx$

$$
\begin{aligned}
&= \lim_{t\to\infty} -\tfrac{1}{2}e^u \,\Big|_{0}^{-t^2}\\
&= \lim_{t\to\infty} -\tfrac{1}{2}e^{-t^2} + \tfrac{1}{2}e^0\\
&= \frac{1}{2} \quad \text{since } e^{-t^2} \to 0 \text{ as } t \to \infty
\end{aligned}
$$

So, $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx = -\tfrac{1}{2} + \tfrac{1}{2} = 0.$

(d) $\displaystyle\int_{e}^{\infty} \frac{\ln x}{x^2}\,dx$

Consider $\displaystyle\lim_{t\to\infty}\int_{e}^{t} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\ln x \,\Big|_{e}^{t} - \int_{e}^{t} -\frac{1}{x^2}\,dx$ Integration by parts: $u = \ln x \;\to\; du = \tfrac{1}{x}\,dx$, $dv = \tfrac{1}{x^2}\,dx \;\to\; v = -\tfrac{1}{x}$

$$
\begin{aligned}
&= \lim_{t\to\infty} -\frac{1}{x}\ln x - \frac{1}{x}\,\Big|_{e}^{t}\\
&= \lim_{t\to\infty} -\frac{1}{t}\ln t - \frac{1}{t} + \frac{1}{e}\ln e + \frac{1}{e}
\end{aligned}
$$

Note: Using L'Hospital's Rule, we have that

$$\lim_{t\to\infty} -\frac{1}{t}\ln t = \lim_{t\to\infty} -\frac{\ln t}{t} = \lim_{t\to\infty} -\frac{\frac{1}{t}}{1} = 0$$

So, $\displaystyle\lim_{t\to\infty}\int_{e}^{t} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{t}\ln t - \frac{1}{t} + \frac{2}{e} = \frac{2}{e}.$

Thus, $\displaystyle\int_{e}^{\infty} \frac{\ln x}{x^2}\,dx$ converges to $\dfrac{2}{e}$.