Math 125
Final Exam Solutions

1. (18 pts.) Evaluate the following limits using l'Hospital's Rule where appropriate. Be sure to justify your answers.

   (a) (9 pts.) $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

   This limit is of the form "∞/∞".

   Using l'Hospitals: $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2(\ln x)\cdot\frac{1}{x}}{1}$

   $= \lim_{x\to\infty} \frac{2\ln x}{x}$

   $= \lim_{x\to\infty} \frac{\frac{2}{x}}{1}$ (Using l'Hospital's again)

   $= 0$

   (b) (9 pts.) $\lim_{t\to 0^+} (\cos t)^{1/t}$

   Let $y = (\cos t)^{1/t} \quad \Rightarrow \quad \ln y = \frac{1}{t}\ln(\cos t)$.

   Consider $\lim_{t\to 0^+} \ln y = \lim_{t\to 0^+} \frac{\ln(\cos t)}{t} = \lim_{t\to 0^+} -\tan t$ (Using l'Hospital's: "0/0")
   $= 0$

   So, $\lim_{t\to 0^+} (\cos t)^{1/t} = \lim_{t\to 0^+} y = \lim_{t\to 0^+} e^{\ln y} = e^0 = 1$.

2. (22 pts.)

   (a) (10 pts.) Approximate the area under the curve of $h(x) = \arctan 2x$ from $x = 0$ to $x = \frac{1}{2}$ using Simpson's Rule with $n = 4$. Give your answer rounded to 4 decimal places.

   $\Delta x = \frac{\frac{1}{2}}{4} = \frac{1}{8}$

   $\Rightarrow$ Area $\approx \frac{1}{8}\cdot\frac{1}{3}[\arctan 0 + 4\arctan(2\cdot\frac{1}{8}) + 2\arctan(2\cdot\frac{1}{4}) + 4\arctan(2\cdot\frac{3}{8}) + \arctan(2\cdot\frac{1}{2})]$

   $\approx .21944$

   (b) (12 pts.) Find the exact area using integrals.

   Area $= \int_0^{1/2} \arctan 2x\, dx$ — Int. by Parts: $u = \arctan 2x, \quad dv = dx$

   $= x\arctan 2x\,\Big|_0^{1/2} - \int_0^{1/2} \frac{2x}{1+4x^2}\, dx$ — $du = \frac{2}{1+(2x)^2}\, dx, \quad v = x$

$= \frac{1}{2}\arctan(2 \cdot \frac{1}{2}) - 0 - \frac{1}{4}\int_1^2 \frac{du}{u}$ Substitution: $u = 1 + 4x^2, \quad du = 8x\ dx$

$= \frac{\pi}{8} - \frac{1}{4}\ln|u|\ |_1^2$

$= \frac{\pi}{8} - \frac{1}{4}\ln\ 2 + \frac{1}{4}\ln\ 1$

$= \frac{\pi}{8} - \frac{1}{4}\ln\ 2 \approx .219412$

3. (10 pts.) Consider $g(x) = \int_0^x e^t \sin(t)\, dt$. Find **all** values of $b > 0$ such that $g'(b) = 0$.

By the FTC, $g'(x) = e^x \sin(x)$. Since e^x is always positive, $g'(b)$ is only equal to zero when $\sin(b) = 0 \quad \Rightarrow \quad g'(b) = 0$ for $b = k\pi$ for an integer k with $k > 0$.

4. (24 pts.) Evaluate each integral or show that it is divergent. Be sure to justify your answers.

(a) (12 pts.) $\int_0^1 \frac{2e^x}{e^x - 1}\, dx$

The integrand is undefined at $x = 0$.

Consider $\lim_{t \to 0^+} \int_t^1 \frac{2e^x}{e^x - 1}\, dx = \lim_{t \to 0^+} 2\int_{e^t - 1}^{e-1} \frac{du}{u}$

Substitution: $u = e^x - 1, \quad du = e^x\ dx$

$$= \lim_{t \to 0^+} 2\ln|u|\ |_{e^t - 1}^{e-1}$$
$$= \lim_{t \to 0^+} 2\ln|e - 1| - 2\ln|e^t - 1|$$
$$= \infty$$

since $e^t - 1 \to 0$ as $t \to 0^+ \quad \Rightarrow \quad \ln|e^t - 1| \to -\infty$

(b) (12 pts.) $\int_1^\infty \frac{-2x - 3}{x^3 + 2x^2 + x}\, dx$ (Recall: $\ln a - \ln b = \ln \frac{a}{b}$)

Note that $\frac{-2x-3}{x^3+2x^2+x} = \frac{-3}{x} + \frac{3}{x+1} + \frac{1}{(x+1)^2}$ by partial fraction decomposition.

Consider $\lim_{t \to \infty} \int_1^t \frac{-3}{x} + \frac{3}{x+1} + \frac{1}{(x+1)^2}\, dx$

$$= \lim_{t \to \infty} -3\ln|x| + 3\ln|x+1| - \frac{1}{x+1}\ |_1^t$$
$$= \lim_{t \to \infty} -3\ln|t| + 3\ln|t+1| - \frac{1}{t+1} + 3\ln\ 1 - 3\ln\ 2 + \frac{1}{2}$$
$$= \lim_{t \to \infty} 3\ln|\frac{t+1}{t}| - \frac{1}{t+1} - 3\ln\ 2 + \frac{1}{2}$$
$$= -3\ln\ 2 + \frac{1}{2}$$

since $\frac{t+1}{t} \to 1$ as $t \to \infty \quad \Rightarrow \quad \ln|\frac{t+1}{t}| \to 0$ as $t \to \infty$.

So, $\displaystyle\int_1^\infty \frac{-2x-3}{x^3+2x^2+x}\,dx = -3\ln 2 + \tfrac{1}{2}$.

5. (14 pts.) The velocity of an annoying mechanical barking dog moving along a straight line is given by the function $v(t) = \frac{2t+4}{\sqrt{t+9}}$ in meters/minute at minute t. Find the function $s(t)$ that gives the position of the annoying mechanical barking dog if the initial position of the dog is 2 meters.

Rationalizing substitution: $u = \sqrt{t+9}, \quad 2u\,du = dt$

Note that
$$\begin{aligned}\int \frac{2t+4}{\sqrt{t+9}}\,dt &= \int \frac{2(u^2-9)+4}{u}\cdot 2u\,du \\ &= \int 4u^2 - 28\,du \\ &= \tfrac{4}{3}u^3 - 28u + C \\ &= \tfrac{4}{3}(t+9)^{3/2} - 28\sqrt{t+9} + C\end{aligned}$$

So, $s(t) = \frac{4}{3}(t+9)^{3/2} - 28\sqrt{t+9} + C$ for some constant C.

Solving for C: $\quad 2 = s(0) = \frac{4}{3}(9)^{3/2} - 28\sqrt{9} + C \quad \Rightarrow \quad C = 50$

$$\Rightarrow \quad s(t) = \tfrac{4}{3}(t+9)^{3/2} - 28\sqrt{t+9} + 50$$