

Math 125
Final Exam Solutions

1. (24 pts.) Evaluate each integral or show that it is divergent. Be sure to justify your answers.

(a) (12 pts.) $\int_1^\infty \frac{3t+2}{t^3+2t^2+t} dt$ (Recall: $\ln a - \ln b = \ln \frac{a}{b}$)

Note that $\frac{3t+2}{t^3+2t^2+t} = \frac{2}{t} + \frac{-2}{t+1} + \frac{1}{(t+1)^2}$ by partial fraction decomposition.

$$\begin{aligned} \text{Consider } \lim_{a \rightarrow \infty} \int_1^a \frac{2}{t} - \frac{2}{t+1} + \frac{1}{(t+1)^2} dt \\ &= \lim_{a \rightarrow \infty} 2\ln|t| - 2\ln|t+1| - \frac{1}{t+1} \Big|_1^a \\ &= \lim_{a \rightarrow \infty} 2\ln|a| - 2\ln|a+1| - \frac{1}{a+1} - 2\ln 1 + 2\ln 2 + \frac{1}{2} \\ &= \lim_{a \rightarrow \infty} 2\ln\left|\frac{a}{a+1}\right| - \frac{1}{a+1} + 2\ln 2 + \frac{1}{2} \\ &= 2\ln 2 + \frac{1}{2} \end{aligned}$$

$$\text{since } \frac{a}{a+1} \rightarrow 1 \text{ as } a \rightarrow \infty \Rightarrow \ln\left|\frac{a}{a+1}\right| \rightarrow 0 \text{ as } a \rightarrow \infty.$$

$$\text{So, } \int_1^\infty \frac{3t+2}{t^3+2t^2+t} dt = 2\ln 2 + \frac{1}{2}.$$

(b) (12 pts.) $\int_0^1 \frac{3e^x}{1-e^x} dx$

The integrand is undefined at $x = 0$.

$$\begin{aligned} \text{Consider } \lim_{t \rightarrow 0^+} \int_t^1 \frac{3e^x}{1-e^x} dx &= \lim_{t \rightarrow 0^+} -3 \int_{1-e^t}^{1-e} \frac{du}{u} && \text{Substitution: } u = 1 - e^x, \quad du = -e^x dx \\ &= \lim_{t \rightarrow 0^+} -3\ln|u| \Big|_{1-e^t}^{1-e} \\ &= \lim_{t \rightarrow 0^+} -3\ln|1-e| + 3\ln|1-e^t| \\ &= -\infty \\ &\text{since } 1 - e^t \rightarrow 0 \text{ as } t \rightarrow 0^+ \Rightarrow \ln|1 - e^t| \rightarrow -\infty \end{aligned}$$

2. (10 pts.) Consider $g(x) = \int_0^x e^t \cos(t) dt$. Find **all** values of $b > 0$ such that $g'(b) = 0$.

By the FTC, $g'(x) = e^x \cos(x)$. Since e^x is always positive, $g'(b)$ is only equal to zero when $\cos(b) = 0 \Rightarrow g'(b) = 0$ for $b = \frac{\pi}{2} + k\pi$ for an integer k with $k \geq 0$.

3. (18 pts.) Evaluate the following limits using l'Hospital's Rule where appropriate. Be sure to justify your answers.

(a) (9 pts.) $\lim_{t \rightarrow \infty} \frac{t}{(\ln t)^2}$

This limit is of the form “ ∞/∞ ”.

Using l'Hospital's:
$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{t}{(\ln t)^2} &= \lim_{t \rightarrow \infty} \frac{1}{2(\ln t) \cdot \frac{1}{t}} \\ &= \lim_{t \rightarrow \infty} \frac{t}{2 \ln t} \\ &= \lim_{t \rightarrow \infty} \frac{1}{\frac{2}{t}} \quad (\text{Using l'Hospital's again}) \\ &= \lim_{t \rightarrow \infty} \frac{t}{2} = \infty \end{aligned}$$

(b) (9 pts.) $\lim_{x \rightarrow 0^+} (\sec x)^{1/x}$

Let $y = (\sec x)^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(\sec x)$.

Consider $\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(\sec x)}{x} = \lim_{x \rightarrow 0^+} \tan x$ (Using l'Hospital's: “0/0”)
 $= 0$

So, $\lim_{x \rightarrow 0^+} (\sec x)^{1/x} = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^0 = 1$.

4. (14 pts.) The velocity of an annoying mechanical barking dog moving along a straight line is given by the function $v(t) = \frac{3t+28}{\sqrt{t+16}}$ in meters/minute at minute t . Find the function $s(t)$ that gives the position of the annoying mechanical barking dog if the initial position of the dog is 2 meters.

Note that $\int \frac{3t+28}{\sqrt{t+16}} dt = \int \frac{3(u^2-16)+28}{u} \cdot 2u du$ Rationalizing substitution:
 $u = \sqrt{t+16}, \quad 2u du = dt$
$$\begin{aligned} &= \int 6u^2 - 40 du \\ &= 2u^3 - 40u + C \\ &= 2(t+16)^{3/2} - 40\sqrt{t+16} + C \end{aligned}$$

So, $s(t) = 2(t+16)^{3/2} - 40\sqrt{t+16} + C$ for some constant C .

Solving for C : $2 = s(0) = 2(16)^{3/2} - 40\sqrt{16} + C \Rightarrow C = 34$

$\Rightarrow s(t) = 2(t+16)^{3/2} - 40\sqrt{t+16} + 34$

5. (22 pts.)

- (a) (10 pts.) Approximate the area under the curve of $f(x) = \arctan 2x$ from $x = 0$ to $x = \frac{1}{2}$ using Simpson's Rule with $n = 4$. Give your answer rounded to 4 decimal

places.

$$\Delta x = \frac{\frac{1}{2}}{4} = \frac{1}{8}$$

$$\begin{aligned}\Rightarrow \text{Area} &\approx \frac{1}{8} \cdot \frac{1}{3} [\arctan 0 + 4\arctan(2 \cdot \frac{1}{8}) + 2\arctan(2 \cdot \frac{1}{4}) + 4\arctan(2 \cdot \frac{3}{8}) + \arctan(2 \cdot \frac{1}{2})] \\ &\approx .21944\end{aligned}$$

(b) (12 pts.) Find the exact area using integrals.

$$\begin{aligned}\text{Area} &= \int_0^{1/2} \arctan 2x \, dx && \text{Int. by Parts: } u = \arctan 2x, \quad dv = dx \\ &= x \arctan 2x \Big|_0^{1/2} - \int_0^{1/2} \frac{2x}{1+4x^2} \, dx && \underline{du = \frac{2}{1+(2x)^2} \, dx, \quad v = x} \\ &= \frac{1}{2} \arctan(2 \cdot \frac{1}{2}) - 0 - \frac{1}{4} \int_1^2 \frac{du}{u} && \text{Substitution: } u = 1 + 4x^2, \quad du = 8x \, dx \\ &= \frac{\pi}{8} - \frac{1}{4} \ln|u| \Big|_1^2 \\ &= \frac{\pi}{8} - \frac{1}{4} \ln 2 + \frac{1}{4} \ln 1 \\ &= \frac{\pi}{8} - \frac{1}{4} \ln 2 \approx .219412\end{aligned}$$