Math 125
Exam 1 Solutions

1. (22 pts.) $g(x) = \frac{42}{x}+1$

(a) (8 pts.) Approximate $\int_1^2 g(x)\,dx$ using 4 right-end rectangles.

Note that $\Delta x = \frac{2-1}{4} = \frac{1}{4}$ and $x_1 = \frac{5}{4},\ x_2 = \frac{3}{2},\ x_3 = \frac{7}{4},\ x_4 = 2.$

$$\begin{aligned}\int_1^2 g(x)\,dx &\approx \tfrac{1}{4}[f(\tfrac{5}{4}) + f(\tfrac{3}{2}) + f(\tfrac{7}{4}) + f(2)] \\ &= \tfrac{1}{4}[\tfrac{168}{5} + 1 + 28 + 1 + 24 + 1 + 21 + 1] \\ &= \tfrac{1}{4}[\tfrac{553}{5}] \\ &= \tfrac{553}{20} = 27.65\end{aligned}$$

(b) (7 pts.) Write the exact value of $\int_1^2 g(x)\,dx$ as the limit of a sum of the areas of right-end rectangles. DO NOT EVALUATE THIS LIMIT.

Using n rectangles, we have $\Delta x = \frac{2-1}{n} = \frac{1}{n}$ and $x_i = 1 + i(\frac{1}{n}) = 1 + \frac{i}{n}$.

So, $\displaystyle\int_1^2 g(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} g(1+\frac{i}{n})\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{42}{1+\frac{i}{n}}+1)\cdot\frac{1}{n}$

(c) (7 pts.) Find the exact value of $\int_1^2 g(x)\,dx$ using the FTC.

$$\begin{aligned}\int_1^2 g(x)\,dx &= 42\ln|x| + x\ |_1^2 \\ &= 42\ln 2 + 2 - (42\ln 1 + 1) \\ &= 42\ln 2 + 1 \\ &\approx 30.1122\end{aligned}$$

2. (22 pts.) Evaluate the following integrals.

(a) (6 pts.) $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ (Hint: Consider the graph of the integrand.)

The graph of $f(x) = \sqrt{9-x^2}$ is the upper half of a circle.
So, $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ = area under the curve f = area of a semicircle of radius 3

$= \frac{1}{2}\pi r^2 = \frac{9}{2}\pi$

(b) (8 pts.) $\int_{-\pi/3}^{0}[\sec(x)\tan(x) + 6x^2]\,dx$

$$\begin{aligned}\int_{-\pi/3}^{0}[\sec(x)\tan(x) + 6x^2]\,dx &= \sec(x) + 2x^3\ |_{-\pi/3}^{0} \\ &= \sec(0) + 2(0)^3 - [\sec(-\tfrac{\pi}{3} + 2(\tfrac{-\pi}{3})^3)]\end{aligned}$$

$= 1 - [2 - \frac{2\pi^3}{27}]$
$= -1 + \frac{2\pi^3}{27}$
≈ 1.2968

(c) (8 pts.) $\int e^{2t}\sqrt{e^{2t}+3}\,dt$

Let $u = e^{2t} + 3$. Then $du = 2e^{2t}\ dt$ or $\frac{1}{2}\ du = e^{2t}\ dt$.

Then $\int e^{2t}\sqrt{e^{2t}+3}\,dt = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2} \cdot \frac{2}{3}u^{3/2} + C$
$= \frac{1}{3}(e^{2t}+3)^{3/2} + C$

3. (14 pts.) Find a function h such that $h'(x) = \frac{1}{3x-2} - \frac{1}{x^2}$ with $h(1) = 0$.

Note that $\int \frac{1}{3x-2}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x-2| + C$.
(Using $u = 3x - 2 \Rightarrow du = 3\ dx$ or $\frac{1}{3}\ du = dx$.)

So, $h(x) = \frac{1}{3}\ln|3x-2| + \frac{1}{x} + C$

To solve for C: $h(1) = \frac{1}{3}\ln|3-2| + 1 + C = 0 \Rightarrow C = -1$

$\Rightarrow h(x) = \frac{1}{3}\ln|3x-2| + \frac{1}{x} - 1$

4. (12 pts.)

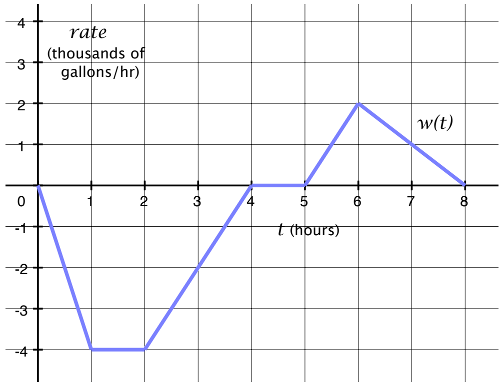
(a) (6 pts.) Write the sum $1 + 4 + 7 + 10 + 13 + 16 + 19$ in sigma notation.

$1 + 4 + 7 + 10 + 13 + 16 + 19 = \sum_{i=0}^{6} 1 + 3i$ or $\sum_{i=1}^{7} 3i - 2$

(b) (6 pts.) Evaluate the following sum: $\sum_{i=0}^{3}[\sin(\frac{\pi}{2} + i\pi) + 3]$

$$\begin{aligned}\sum_{i=0}^{3}[\sin(\frac{\pi}{2}+i\pi)+3] &= \sin(\frac{\pi}{2})+3+\sin(\frac{\pi}{2}+\pi)+3+\sin(\frac{\pi}{2}+2\pi)+3+\sin(\frac{\pi}{2}+3\pi)+3 \\ &= 1+3-1+3+1+3-1+3 \\ &= 12\end{aligned}$$

5. (18 pts.) The following graph of $w(t)$ gives the **rate of change** (in thousands of gallons/hour) of the volume of water in a reservoir at hour t.



(a) (7 pts.) What are the units of $\int_0^8 w(t)\,dt$ and what does the integral represent in terms of water volume?

The units of $\int_0^8 w(t)\,dt$ will be thousands of gallons.

The integral represents the net change in the volume of water in the reservoir between $t = 0$ and $t = 8$ hours.

(b) (7 pts.) Find the exact value of $\int_0^8 w(t)\,dt$.

We can calculate the exact value of $\int_0^8 w(t)\,dt$ using areas.

The area of above the curve (below the x-axis) between $t = 0$ and $t = 4$ is 10. The area below the curve (above the x-axis) between $t = 5$ and $t = 8$ is 3.

So, $\int_0^8 w(t)\,dt = 3 - 10 = -7$ thousand gallons.

(c) (4 pts.) When is the water level the highest during the interval $0 \le t \le 8$?

The water level will be at its highest at $t = 0$ since the volume of water decreases from $t = 0$ to $t = 4$ and then increases again from $t = 5$ and $t = 8$. The amount of water that flows out of the reservoir in the 8 hours (10 thousand gallons) is more than the amount that flows into the reservoir (3 thousand gallons.

6. (12 pts.) Find a function f such that $\int_1^x [t^2 f(t) + 2]\,dt = -x^5 + 2x - 1$.

Taking the derivative of both sides of the equation with respect to x:

$$x^2 f(x) + 2 = -5x^4 + 2 \quad \text{(Using FTC part 1.)}$$

Solving for $f(x)$: $x^2 f(x) = -5x^4 \quad \Rightarrow \quad f(x) = -5x^2$