

**Math 125**  
**Final Exam Solutions**

1. (10 pts.)

- (a) (5 pts.) Set up the integral used to find the length of the curve  $y = 4x^{1/2}$  from  $x = 1$  to  $x = 4$ . Do not evaluate.

$$\frac{dy}{dx} = 2x^{-1/2} = \frac{2}{\sqrt{x}}. \text{ So } L = \int_1^4 \sqrt{1 + \left(\frac{2}{\sqrt{x}}\right)^2} dx = \int_1^4 \sqrt{1 + \frac{4}{x}} dx$$

- (b) (5 pts.) Set up the sum to approximate the arclength from part *a* using Simpson's Rule with  $n = 6$ . Do not evaluate.

$$\begin{aligned} \Delta x &= \frac{4-1}{6} = \frac{1}{2} \\ \Rightarrow S_6 &= \frac{1}{6}[\sqrt{1+4} + 4\sqrt{1+\frac{4}{1.5}} + 2\sqrt{1+2} + 4\sqrt{1+\frac{4}{2.5}} + 2\sqrt{1+\frac{4}{3}} + 4\sqrt{1+\frac{4}{3.5}} + \sqrt{1+1}] \end{aligned}$$

2. (10 pts.) Evaluate  $\int \sec^3 \theta \tan^3 \theta d\theta$

$$\begin{aligned} \int \sec^3 \theta \tan^3 \theta d\theta &= \int \sec^3 \theta \tan^2 \theta \sec \theta \tan \theta d\theta = \int \sec^2 \theta (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta \\ &= \int u^2(u^2 - 1) du \quad (u = \sec \theta \Rightarrow du = \sec \theta \tan \theta) \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 \theta - \frac{1}{3}\sec^3 \theta + C \end{aligned}$$

3. (5 pts.) For  $g(x) = \int_1^{e^x} [\sin(t^2) + 1] dt$ , what is  $g'(x)$ ?

$$g'(x) = (\sin(e^{2x}) + 1) \cdot \frac{d}{dx}[e^x] = e^x(\sin(e^{2x}) + 1)$$

4. (25 pts.) Evaluate the following integrals.

(a) (10 pts.)  $\int_0^8 \frac{1}{\sqrt[3]{x-8}} dx$

$$\begin{aligned} \int_0^8 \frac{1}{\sqrt[3]{x-8}} dx &= \lim_{t \rightarrow 8^-} \int_0^t \frac{1}{\sqrt[3]{x-8}} dx = \lim_{t \rightarrow 8^-} \frac{3}{2}(x-8)^{2/3} \Big|_0^t \\ &= \lim_{t \rightarrow 8^-} \frac{3}{2}[(t-8)^{2/3} - (0-8)^{2/3}] \\ &= \frac{3}{2}(-4) = -6 \end{aligned}$$

(b) (15 pts.)  $\int_2^\infty \frac{3}{x^2 - x} dx$  Hint: Recall that  $\ln(a) - \ln(b) = \ln(\frac{a}{b})$ .

$$\begin{aligned} \int_2^\infty \frac{3}{x^2 - x} dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{3}{x^2 - x} dx \\ &= \lim_{t \rightarrow \infty} \int_2^t \frac{-3}{x} + \frac{3}{x-1} dx \quad (\text{Using partial fraction decomposition on } \frac{3}{x(x-1)}) \\ &= \lim_{t \rightarrow \infty} -3\ln|x| + 3\ln|x-1| \Big|_2^t \\ &= \lim_{t \rightarrow \infty} 3\ln\left|\frac{x-1}{x}\right| \Big|_2^t \\ &= \lim_{t \rightarrow \infty} 3\ln\left|\frac{t-1}{t}\right| - 3\ln\left|\frac{2-1}{2}\right| \\ &= -3\ln\left(\frac{1}{2}\right) = 3\ln 2 \approx 2.0794 \end{aligned}$$

since as  $t \rightarrow \infty$ ,  $\frac{t-1}{t} \rightarrow 1 \Rightarrow \ln\left|\frac{t-1}{t}\right| \rightarrow 0$ .

5. (15 pts.) A particle moves along a straight line with velocity given by  $v(t) = \frac{t+1}{\sqrt{4-t^2}}$  in nanometers per second. What is the distance traveled in nanometers from  $t = 0$  to  $t = 1$  seconds?

$$\text{Distance traveled} = \int_0^1 |v(t)| dt = \int_0^1 \left| \frac{t+1}{\sqrt{4-t^2}} \right| dt = \int_0^1 \frac{t+1}{\sqrt{4-t^2}} dt$$

Trigonometric substitution:  $t = 2\sin \theta \Rightarrow dt = 2\cos \theta d\theta$

$$\begin{aligned} \int_0^1 \frac{t+1}{\sqrt{4-t^2}} dt &= \int_0^{\pi/6} \frac{2\sin \theta + 1}{\sqrt{4-4\sin^2 \theta}} 2\cos \theta d\theta = \int_0^{\pi/6} 2\sin \theta + 1 d\theta \\ &= -2\cos \theta + \theta \Big|_0^{\pi/6} = -2\cos(\pi/6) + \pi/6 - (-2\cos(0) + 0) \\ &= -\sqrt{3} + \pi/6 + 2 \approx .7915 \end{aligned}$$

6. (20 pts.)

(a) (10 pts.) For what value(s) of  $k$  is  $y = \sqrt[3]{kx^2 + C}$  a solution of  $y' = \frac{x}{y^2}$ ?

$$\text{Since } y = \sqrt[3]{kx^2 + C}, y' = \frac{1}{3}(kx^2 + C)^{-2/3}(2kx) = \frac{2}{3}k \frac{x}{(kx^2 + C)^{2/3}} = \frac{2}{3}k \frac{x}{y^2}$$

We want  $y' = \frac{x}{y^2}$ , so we need  $\frac{2}{3}k \frac{x}{y^2} = \frac{x}{y^2} \Rightarrow \frac{2}{3}k = 1 \Rightarrow k = \frac{3}{2}$

- (b) (10 pts.) Find the solution  $y = f(x)$  for the differential equation  $\frac{dy}{dx} - \frac{\sqrt{y}}{1+x^2} = 0$  with the initial condition  $f(0) = 4$ .

$$\begin{aligned}\frac{dy}{dx} - \frac{\sqrt{y}}{1+x^2} = 0 &\Rightarrow \frac{dy}{dx} = \frac{\sqrt{y}}{1+x^2} \Rightarrow y^{-1/2} dy = \frac{1}{1+x^2} dx \\ &\Rightarrow \int y^{-1/2} dy = \int \frac{1}{1+x^2} dx \\ &\Rightarrow 2y^{1/2} = \arctan x + C\end{aligned}$$

Plugging in the IC:  $x = 0, y = 4 \quad 4^{1/2} = \pm 2 \Rightarrow 2(\pm 2) = \arctan 0 + C = C$   
So  $C = \pm 4$ .

$$\Rightarrow 2y^{1/2} = \arctan x \pm 4 \Rightarrow y = \left(\frac{1}{2}\arctan x \pm 2\right)^2$$

7. (15 pts.) Let  $R$  be the region bounded by  $y = \sin^2 x$  and the  $x$ -axis for  $0 \leq x \leq 2\pi$ . Find the volume of the solid obtained by revolving  $R$  about the  $y$ -axis.

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi} 2\pi x \sin^2 x dx = 2\pi \int_0^{2\pi} x \sin^2 x dx = 2\pi \int_0^{2\pi} x \frac{1}{2}(1 - \cos(2x)) dx \\ &= \pi \int_0^{2\pi} x - x \cos(2x) dx \\ &= \pi \left[ \frac{1}{2}x^2 \Big|_0^{2\pi} - \left( \frac{1}{2}x \sin(2x) \Big|_0^{2\pi} - \frac{1}{4}\cos(2x) \Big|_0^{2\pi} \right) \right] \quad (\text{Integration by Parts: } u = x, dv = \cos(2x) dx) \\ &= 2\pi^3\end{aligned}$$