Math 125 Quiz 4 Solutions

1. Using cylindrical shells, find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = sin(3x)$, $y = 0$, $x = \frac{2\pi}{3}$, and $x = \pi$.

   Drawing the figure and looking at a cross section of the original region at x ($\frac{2\pi}{3} \leq x \leq \pi$), we have that the cylinder generated by rotating this cross section about the y-axis has radius x and height $sin(3x)$. $\Rightarrow A(x) = 2\pi x sin(3x)$

   So, the volume of the solid is
   $Volume = \int_{\frac{2\pi}{3}}^{\pi} A(x)\,dx = \int_{\frac{2\pi}{3}}^{\pi} 2\pi x sin(3x)\,dx = 2\pi \int_{\frac{2\pi}{3}}^{\pi} x sin(3x)\,dx$

   Using integration by parts with $u = x$ and $dv = sin(3x)dx$ ($\Rightarrow du = dx$ and $v = \frac{-cos(3x)}{3}$), we have
   $Volume = 2\pi[-\frac{1}{3}x cos(3x)|_{\frac{2\pi}{3}}^{\pi} - \int_{\frac{2\pi}{3}}^{\pi} -\frac{1}{3}cos(3x)\,dx]$
   $= 2\pi[(-\frac{1}{3}\pi cos(3\pi) + \frac{1}{3}(\frac{2\pi}{3})cos(3 \cdot \frac{2\pi}{3})) + \frac{1}{3}\int_{\frac{2\pi}{3}}^{\pi} cos(3x)\,dx]$
   $= 2\pi[(\frac{\pi}{3} + \frac{2\pi}{9}) + \frac{1}{3} \cdot \frac{1}{3}sin(3x)|_{\frac{2\pi}{3}}^{\pi}]$

   $= 2\pi[\frac{5\pi}{9} + \frac{1}{9}(sin(3\pi) - sin(3 \cdot \frac{2\pi}{3}))]$

   $= 2\pi[\frac{5\pi}{9} + 0] = \frac{10\pi^2}{9}$

2. The velocity of a snail is given by $v(t) = \frac{1}{2}arctan(t) = \frac{1}{2}tan^{-1}(t)$ feet per minute where t is in minutes.

   (a) What is the average velocity of the snail from $t = 0$ to $t = 10$ minutes?
   $v_{ave} = \frac{1}{10-0}\int_0^{10} \frac{1}{2}arctan(t)\,dt = \frac{1}{20}\int_0^{10} arctan(t)\,dt$

   Using integration by parts with $u = arctan(t)$ and $dv = dt$ ($\Rightarrow du = \frac{1}{1+t^2}dt$ and $v = t$), we have
   $v_{ave} = \frac{1}{20}[t \cdot arctan(t)|_0^{10} - \int_0^{10} \frac{t}{1+t^2}\,dt]$

   Using a u-substitution with $u = 1 + t^2$ ($du = 2tdt$ or $\frac{du}{2} = dt$), we have
   $v_{ave} = \frac{1}{20}[(10 \cdot arctan(10) - 0) - \frac{1}{2}\int_1^{101} \frac{1}{u}\,du]$

   $= \frac{1}{20}[14.7112 - \frac{1}{2}ln|u||_1^{101}]$

   $= \frac{1}{20}[14.7112 - \frac{1}{2}(4.6151 - 0)] = \frac{1}{20}[12.4037] = .62019$

(b) As $b \to \infty$, what does the average velocity of the snail from $t = 0$ to $t = b$ approach?

Two Methods:

i. $v_{ave} = \frac{1}{b-0} \int_0^b \frac{1}{2} arctan(t)\, dt = \frac{1}{2b} \int_0^b arctan(t)\, dt$

Using integration by parts with $u = arctan(t)$ and $dv = dt$ ($\Rightarrow du = \frac{1}{1+t^2} dt$ and $v = t$), we have

$v_{ave} = \frac{1}{2b}[t \cdot arctan(t)|_0^b - \int_0^b \frac{t}{1+t^2}\, dt]$

Using a u-substitution with $u = 1 + t^2$ ($du = 2tdt$ or $\frac{du}{2} = dt$), we have

$v_{ave} = \frac{1}{2b}[(b \cdot arctan(b) - 0) - \frac{1}{2} \int_1^{1+b^2} \frac{1}{u}\, du]$

$= \frac{1}{2} arctan(b) - \frac{1}{4b} ln|u||_1^{1+b^2}]$

$= \frac{1}{2} arctan(b) - \frac{1}{4b} ln(1 + b^2)$

So $\lim_{b\to\infty} v_{ave} = \lim_{b\to\infty}[\frac{1}{2} arctan(b) - \frac{1}{4b} ln(1 + b^2)]$

$= \lim_{b\to\infty} \frac{1}{2} arctan(b) - \lim_{b\to\infty} \frac{1}{4b} ln(1 + b^2)$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

since $\lim_{b\to\infty} arctan(b) = \frac{\pi}{2}$ and the second limit is 0 (use L'Hospital's Rule).

ii. Looking at the graph of $\frac{1}{2} arctan(t)$, one can see that as $b \to \infty$, the average value is going to approach the horizontal asymptote $y = \frac{\pi}{4}$.

So $\lim_{b\to\infty} v_{ave} = \frac{\pi}{4}$.