Math 125
Midterm 2 Solutions

1. (20 pts.) What is the average value of the function $f(x) = \frac{-2(x+5)}{x^2-2x-8}$ on the interval $[0,3]$?

$f_{ave} = \frac{1}{3}\int_0^3 \frac{-2(x+5)}{x^2-2x-8}\,dx$

$x^2-2x-8 = (x-4)(x+2) \;\Rightarrow\; \frac{-2(x+5)}{x^2-2x-8} = \frac{A}{x-4} + \frac{B}{x+2}$

$\Rightarrow\; -2(x+5) = A(x+2) + B(x-4)$ Using the values $x=4$ and $x=-2$, we have that $A=-3$ and $B=1$.

$$\begin{aligned}\text{So, } f_{ave} = \tfrac{1}{3}\int_0^3 \frac{-2(x+5)}{x^2-2x-8}\,dx &= \frac{1}{3}\int_0^3 \frac{-3}{x-4} + \frac{1}{x+2}\,dx \\ &= \tfrac{1}{3}[-3ln|x-4| + ln|x+2|]_0^3 \\ &= \tfrac{1}{3}[-3ln(1) + ln(5) - (-3ln(4) + ln(2))] \\ &= \tfrac{1}{3}[ln(5) + 6ln(2) - ln(2)] \\ &= \tfrac{1}{3}[ln(5) + 5ln(2)] \approx 1.6917\end{aligned}$$

2. (25 pts.) Evaluate the following integrals.

(a) (10 pts.) $\int \sin x \ln(\sec x)\,dx$

Using integration by parts with $u = \ln(\sec x)$ and $dv = \sin x\,dx$ ($du = \frac{1}{\sec x}(\sec x \tan x)dx = \tan x\,dx$, $v = -\cos x$), we have

$$\begin{aligned}\int \sin x \ln(\sec x)\,dx &= -\cos x \ln(\sec x) - \int -\cos x \tan x\,dx \\ &= -\cos x \ln(\sec x) + \int \sin x\,dx \\ &= -\cos x \ln(\sec x) - \cos x + C\end{aligned}$$

(b) (15 pts.) $\int \frac{x}{\sqrt{x^2-6x+5}}\,dx$

Completing the square we have $x^2-6x+5 = (x-3)^2 - 9 + 5 = (x-3)^2 - 4$

$$\int \frac{x}{\sqrt{x^2-6x+5}}\,dx = \int \frac{x}{\sqrt{(x-3)^2-4}}\,dx$$

Using the trigonometric substitution $x-3 = 2\sec\theta$ ($dx = \sec\theta\tan\theta\,d\theta$), we have

$$
\begin{aligned}
\int \frac{x}{\sqrt{(x-3)^2-4}}\,dx &= \int \frac{2\sec\,\theta+3}{\sqrt{(2\sec\,\theta)^2-4}} 2\sec\,\theta\,\tan\,\theta\,d\theta \\
&= \int \frac{2\sec\,\theta+3}{\sqrt{4\tan^2\,\theta}} 2\sec\,\theta\,\tan\,\theta\,d\theta \\
&= \int (2\sec\,\theta+3)\sec\,\theta\,d\theta \\
&= \int 2\sec^2\,\theta + 3\sec\,\theta\,d\theta \\
&= 2\tan\,\theta + 3\ln(\sec\,\theta+\tan\,\theta)+C
\end{aligned}
$$

Using a reference triangle, we see that since $\sec\,\theta = \frac{x-3}{2}$, we have $\tan\,\theta = \frac{\sqrt{(x-3)^2-4}}{2}$.

So, $\displaystyle\int \frac{x}{\sqrt{(x-3)^2-4}}\,dx = \sqrt{(x-3)^2-4} + 3\ln(\frac{x-3}{2} + \frac{\sqrt{(x-3)^2-4}}{2}) + C.$

3. (15 pts.) Let R be the region enclosed by $y = \sqrt[3]{x+1}$ and the x-axis for $0 \le x \le 7$. Using cylindrical shells, find the volume of the solid obtained by revolving the region R about the y-axis.

Using cylindrical shells, we have that

$Volume = 2\pi \displaystyle\int_0^7 x\ \sqrt[3]{x+1}\,dx.$

With the substitution $u = \sqrt[3]{x+1}$ ($u^3 = x+1 \ \Rightarrow\ 3u^2\ du = dx$, $x = 0\ \rightarrow\ u = 1$, $x = 7\ \rightarrow\ u = 2$), we have

$$
\begin{aligned}
Volume = 2\pi \int_0^7 x\ \sqrt[3]{x+1}\,dx &= 2\pi \int_1^2 (u^3-1)u \cdot 3u^2\,du \\
&= 6\pi \int_1^2 (u^3-1)u^3\,du \\
&= 6\pi \int_1^2 u^6 - u^3\,du \\
&= 6\pi[\tfrac{1}{7}u^7 - \tfrac{1}{4}u^4]_1^2 \\
&= 6\pi[\tfrac{2^7}{7} - \tfrac{2^4}{4} - (\tfrac{1}{7} - \tfrac{1}{4})] \\
&= 6\pi[\tfrac{127}{7} - \tfrac{15}{4}] = \tfrac{1209}{14}\pi \approx 271.2990
\end{aligned}
$$