Math 125
Midterm 1 Solutions

1. (15 pts.) Evaluate the following.

   (a) (5 pts.) For $x > 0$, if $g(x) = \int_3^{ln(x)} [t^2 + 4sin(t)]\, dt$, what is $g'(x)$?

   Using the FTC, we have that $g'(x) = [ln^2(x) + 4sin(ln(x))] \cdot \frac{d}{dx}[ln(x)]$

   $$= \frac{1}{x}[ln^2(x) + 4sin(ln(x))]$$

   (b) (5 pts.) $\int \frac{x^2}{1+x^6}\, dx = ?$

   Using u-substitution with $u = x^3$, we have $du = 3x^2\ dx$ or $\frac{du}{3} = x^2\ dx$.
   So, $\int \frac{x^2}{1+x^6}\, dx = \frac{1}{3}\int \frac{1}{1+u^2}\, du = \frac{1}{3}tan^{-1}(u) + C = \frac{1}{3}tan^{-1}(x^3) + C$

   (c) (5 pts.) $\int_{-\pi/2}^{\pi} [cos(t) + \frac{3}{\pi}]\, dt$

   $$\int_{-\pi/2}^{\pi} [cos(t) + \frac{3}{\pi}]\, dt = [sin(t) + \frac{3}{\pi}t]_{-\pi/2}^{\pi} = [sin(\pi) + \frac{3}{\pi}\pi] - [sin(-\pi/2) + \frac{3}{\pi}(-\pi/2)]$$
   $$= [0 + 3] - [1 - \tfrac{3}{2}] = 3.5$$

2. (30 pts.) $f(x) = 3sin(\frac{1}{2}x)$

   (a) (7 pts.) Estimate the area of the region under the graph of f between $x = 0$ and $x = 2\pi$ using 4 approximating right-end rectangles.

   Using 4 right-end rectangles, we have that $\Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}$
   $\Rightarrow\ x_1 = \frac{\pi}{2},\ x_2 = \pi,\ x_3 = \frac{3\pi}{2}$, and $x_4 = 2\pi$.

   So, Area $\approx \sum_{i=1}^{4} 3sin(\frac{1}{2}x_i)\Delta x$

   $= 3sin(\frac{1}{2}\cdot\frac{\pi}{2})\cdot\frac{\pi}{2} + 3sin(\frac{1}{2}\cdot\pi)\cdot\frac{\pi}{2} + 3sin(\frac{1}{2}\cdot\frac{3\pi}{2})\cdot\frac{\pi}{2} + 3sin(\frac{1}{2}\cdot 2\pi)\cdot\frac{\pi}{2}$

   $= \frac{\pi}{2}[3sin(\frac{\pi}{4}) + 3sin(\frac{\pi}{2}) + 3sin(\frac{3\pi}{4}) + 3sin(\pi)] = \frac{\pi}{2}[3\cdot\frac{\sqrt{2}}{2} + 3\cdot 1 + 3\cdot\frac{\sqrt{2}}{2} + 3\cdot 0]$

   $= \frac{\pi}{2}[3\sqrt{2} + 3] \approx 11.3767$

   (b) (5 pts.) Write the exact area under the graph of f between $x = 0$ and $x = 2\pi$ as a limit of a sum of approximating right-end rectangles. DO NOT EVALUATE THIS LIMIT.

Given n right-end rectangles, we have that $\Delta x = \frac{2\pi - 0}{n} = \frac{2\pi}{n}$
$\Rightarrow \ x_i = i\frac{2\pi}{n} = \frac{2\pi i}{n}$.

So, Area $= \sum_{i=1}^{n} 3sin(\frac{1}{2}x_i)\Delta x = \sum_{i=1}^{4} 3sin(\frac{1}{2} \cdot \frac{2\pi i}{n})\frac{2\pi}{n} = \sum_{i=1}^{4} 3sin(\frac{\pi i}{n})\frac{2\pi}{n}$

(c) (8 pts.) Find the exact area under the graph of f between $x = 0$ and $x = 2\pi$ using integration.

Area $= \int_0^{2\pi} 3sin(\frac{1}{2}x)\, dx$

Using u-substitution with $u = \frac{x}{2}$, we have $du = \frac{dx}{2}$ or $2du = dx$.

So, $\int_0^{2\pi} 3sin(\frac{1}{2}x)\, dx = 2\int_0^{\pi} 3sin(u)\, du = 6\int_0^{\pi} sin(u)\, du$
$= 6[-cos(x)]_0^{\pi} = -6[cos(\pi) - cos(0)] = -6(-2) = 12$

(Limits of Integration: $x = 0 \ \Rightarrow \ u = 0$ and $x = 2\pi \ \Rightarrow \ u = \pi$)

(d) (10 pts.) What is the area between $f(x)$ and $g(x) = sin(x)$ between $x = \pi$ and $x = 3\pi$?

Area between curves $= \int_{\pi}^{3\pi} |3sin(\frac{1}{2}x) - sin(x)|\, dx$

Note that g(x) is below f(x) from $x = \pi$ to $x = 2\pi$ and g(x) is above f(x) from $x = 2\pi$ to $x = 3\pi$.

So, $\int_{\pi}^{3\pi} |3sin(\frac{1}{2}x) - sin(x)|\, dx = \int_{\pi}^{2\pi} 3sin(\frac{1}{2}x) - sin(x)\, dx + \int_{2\pi}^{3\pi} sin(x) - 3sin(\frac{1}{2}x)\, dx$
$= [-6cos(\frac{x}{2}) + cos(x)]_{\pi}^{2\pi} + [-cos(x) + 6cos(\frac{x}{2})]_{2\pi}^{3\pi}$
$= [-6cos(\pi) + cos(2\pi) - (-6cos(\frac{\pi}{2}) + cos(\pi))] + [-cos(3\pi) + 6cos(\frac{3\pi}{2}) - (-cos(2\pi) + 6cos(\pi))]$
$= [6 + 1 - (0 - 1)] + [1 + 0 - (-1 - 6)] = 8 + 8 = 16$

3. (15 pts.) Let R be the region enclosed by the graphs of $y = e^x$, $y = e^2$, and the y-axis. Find the volume of the solid obtained by revolving the region R about the x-axis.

Since we are revolving around the x-axis, we will want to cut cross-sections perpendicular to the x-axis to use the washer method. Note that $y = e^2$ and $y = e^x$ intersect at $x = 2$, so we will be integrating from $x = 0$ to $x = 2$.

Using washers, at a given x so that $0 \le x \le 2$, we have that the outer disk has radius e^2 and the inner disk has radius e^x. So the area of a cross-section at x is $A(x) = \pi[(e^2)^2 - (e^x)^2] = \pi[e^4 - e^{2x}]$.

So, Volume $= \int_0^2 \pi[e^4 - e^{2x}]\, dx = \pi\int_0^2 [e^4 - e^{2x}]\, dx = \pi[e^4 x - \frac{1}{2}e^{2x}]_0^2$

$= \pi[2e^4 - \frac{1}{2}e^4 - (0 - \frac{1}{2}e^0)] = \pi[\frac{3}{2}e^4 + \frac{1}{2}] \approx 258.8585$