

Math 124
Final Exam Solutions

1. (a) Using the chain rule and the quotient rule:

$$\begin{aligned} f'(x) &= 12(\ln x \cdot e^{\tan x})^{11} \cdot \frac{d}{dx}[\ln x \cdot e^{\tan x}] + \sqrt{2} \cdot 3x^2 \\ &= 12(\ln x \cdot e^{\tan x})^{11} \cdot \left(\frac{1}{x} \cdot e^{\tan x} + \ln x \cdot e^{\tan x} \cdot \sec^2 x\right) + 3\sqrt{2}x^2 \end{aligned}$$

- (b) Using the quotient rule and the chain rule for the derivative of the numerator:

$$\begin{aligned} \frac{d}{dt} \left[\frac{\sqrt{\arctan(5t)}}{6t^2+8} - 3t \right] &= \frac{(6t^2+8) \cdot \frac{d}{dt}[\sqrt{\arctan(5t)}] - \sqrt{\arctan(5t)} \cdot \frac{d}{dt}[6t^2+8]}{(6t^2+8)^2} - 3 \\ &= \frac{(6t^2+8) \cdot \frac{1}{2}(\arctan(5t))^{-1/2} \cdot \frac{1}{1+(5t)^2} \cdot 5 - \sqrt{\arctan(5t)} \cdot 12t}{(6t^2+8)^2} - 3 \end{aligned}$$

2. We have that $h(2) = 7$ and $h'(2) = -3$. Since 2:15 pm corresponds to $t = 2.25$, we want to use linear approximation to estimate $h(2.25)$.

$$\begin{aligned} \text{Linear Approximation: } h(2.25) &\approx h'(2)(2.25 - 2) + h(2) \\ &= -3(0.25) + 7 \\ &= 6.25 \text{ feet} \end{aligned}$$

So, the height of the tide is approximately 6.25 feet at 2:15 pm.

3. (a) The function $f(x)$ is decreasing when $f'(x)$ is negative. $\Rightarrow f(x)$ is decreasing on $(-\infty, -2)$ or $(-2, 1)$. This can also be written as $x < -2$ or $-2 < x < 1$.
- (b) The critical numbers of $f(x)$ are $x = -2$ and $x = 1$, since those are the zeros of $f'(x)$ and there are no values for which $f'(x)$ is undefined.

Since $f'(x)$ does not change sign at $x = -2$, there is neither a local maximum nor a local minimum at $x = -2$.

Since $f'(x)$ switches from negative to positive at $x = 1$, $f(x)$ has a local minimum at $x = 1$.

- (c) The function $f(x)$ is concave up when $f''(x)$ is positive. Since $f''(x)$ gives the slope of $f'(x)$, $f(x)$ is concave up when the slope of $f'(x)$ is positive. $\Rightarrow f(x)$ is concave up on $(-\infty, -2)$ or $(0, 2)$.

The function $f(x)$ is concave down when $f''(x)$ is negative. Since $f''(x)$ gives the slope of $f'(x)$, $f(x)$ is concave down when the slope of $f'(x)$ is negative. $\Rightarrow f(x)$ is concave down on $(-2, 0)$ or $(2, \infty)$.

4. (a) Let V = volume of the water in the tank,
 h = height of the water (water level), and

r = radius of the water surface.

Given: $\frac{dV}{dt} = 2 \text{ m}^3/\text{min}$ **Unknown:** $\frac{dh}{dt}$ when $h = 4\text{m}$

Equation: $V = \frac{1}{3}\pi r^2 h$

Since we do not have information about $\frac{dr}{dt}$, we will eliminate r from the equation using similar triangles. Note that by looking at the dimensions of the tank and comparing them with r and h , we have that $\frac{r}{h} = \frac{3}{12}$ or $r = \frac{1}{4}h$.

Modified Equation: $V = \frac{1}{3}\pi(\frac{1}{4}r)^2 h = \frac{1}{48}\pi h^3$

Differentiating both sides with respect to t :

$$\frac{dV}{dt} = \frac{1}{48}\pi(3h^2)\frac{dh}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt}$$

Plugging in known values to solve for $\frac{dh}{dt}$:

$$2 = \frac{1}{16}\pi(4)^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{2}{\pi} \text{ m/min}$$

So, the water level rises $\frac{2}{\pi}$ m/min.

(b) The volume of the tank is $\frac{1}{3}\pi(3)^2(12) = 36\pi$ meters³.

Since it fills up at a rate of $2 \text{ m}^3/\text{min}$, it will take $\frac{36\pi}{2} = 18\pi \approx 56.5487$ minutes.

5. The point of tangency is given by $(x(\frac{\pi}{2}), y(\frac{\pi}{2})) = (1, 6)$.

Note that $x'(t) = -6 \sin t + \sin(6t) \cdot 6$ and $y'(t) = 6 \cos t - \cos(6t) \cdot 6$.

So, $x'(\frac{\pi}{2}) = -6$ and $y'(\frac{\pi}{2}) = 6$.

The slope of the curve when $t = \frac{\pi}{2}$ is $\frac{dy}{dx} \Big|_{t=\pi/2} = \frac{y'(\frac{\pi}{2})}{x'(\frac{\pi}{2})} = \frac{6}{-6} = -1$.

Thus, the equation of the tangent line is $y - 6 = -1(x - 1)$ or $y = -x + 7$.

6. (a) Derivative: $f'(x) = 6x^2 \cdot e^x + 2x^3 \cdot e^x$ (Product Rule)
 $= 2x^2 e^x (3 + x)$

Note that $f'(x)$ is defined for all real values.

Critical Numbers: $f'(x) = 0$ when $2x^2 e^x (3 + x) = 0$

Since $e^x > 0$ for all values, $f'(x) = 0$ when $2x^2(3 + x) = 0 \Rightarrow x = 0, x = -3$

So, the critical numbers are $x = 0$ and $x = -3$.

• **First Derivative Test:**

Sign Chart:

f'	-	-	-	-	0	+	+	+	+	0	+	+	+	+
x					-3					0				

Plugging in x -values less than -3 , between -3 and 0 , and great than 0 into f' will yield the above sign chart.

From the sign of the derivative we can see that f is increasing for the values $-3 < x < 0$ and $x > 0$. The function f is decreasing for $x < -3$.

So, there is a local min at $x = -3$ and $f(x)$ has neither a local max nor a local min at $x = 0$.

• **Second Derivative Test:**

$$\begin{aligned} f''(x) &= 12x \cdot e^x + 6x^2 \cdot e^x + 6x^2 \cdot e^x + 2x^3 \cdot e^x \\ &= 12x \cdot e^x + 12x^2 \cdot e^x + 2x^3 \cdot e^x \end{aligned}$$

Plugging in critical numbers: $f''(-3) \approx 0.8962 > 0 \Rightarrow$ Local min at $x = -3$
 $f''(0) = 0 \Rightarrow$ Inconclusive (Must use 1st deriv. test)

So, there is a local min at $x = -3$ and $f(x)$ has neither a local max nor a local min at $x = 0$.

(b) In the domain of $[-4, 1]$, we have the critical numbers $x = 0$ and $x = -3$.

Since the absolute maximum or minimum values can only occur at critical numbers or end-points, we can find the values by evaluating the function at $x = 0$, $x = -3$, $x = -4$, and $x = 1$.

$$\begin{aligned} f(0) &= 0 & f(-4) &= 2(-4)^3 e^{-4} = -128e^{-4} \approx -2.3444 \\ f(-3) &= 2(-3)^3 \cdot e^{-3} = -54e^{-3} \approx -2.6885 & f(1) &= 2(1)^3 e^1 = 2e \approx 5.4366 \end{aligned}$$

So, the absolute maximum value is $2e$ and it occurs at $x = 1$.
The absolute minimum value is $-54e^{-3}$ and it occurs at $x = -3$.

(Note: It is unnecessary to evaluate at $x = 0$ since from part (a), we know that there is no local max and min at $x = 0$.)

7. (a) Since we are considering values of x such that $x > 1$, we have that $x-1 > 0 \Rightarrow |x-1| = x-1$.

$$\text{So, } \lim_{x \rightarrow 1^+} \frac{4|x-1|}{x^2-1} = \lim_{x \rightarrow 1^+} \frac{4(x-1)}{(x-1)(x+1)} = \lim_{x \rightarrow 1^+} \frac{4}{x+1} = \frac{4}{1+1} = 2.$$

(b) Note that this limit is of the indeterminate form $\frac{0}{0}$. Using L'Hospital's Rule, we have that

$$\lim_{x \rightarrow 1} \frac{3x-3}{\ln x} = \lim_{x \rightarrow 1} \frac{3}{\frac{1}{x}} = \lim_{x \rightarrow 1} 3x = 3.$$

- (c) Note that as t approaches 2 from the left, the numerator e^t approaches e^2 and the denominator approaches 0. \Rightarrow This limit is infinite.

Since the numerator is approaching a positive number (e^2) and the denominator is negative ($t - 2 < 0 \Rightarrow (t - 2)^3 < 0$) and approaching 0, this function is approaching $-\infty$.

$$\text{So, } \lim_{t \rightarrow 2^-} \frac{e^t}{(t - 2)^3} = -\infty$$

8. (a) Note that e^x is continuous for all values.

Also, note that $\frac{x^2 - 16}{3x^2 + 12x} = \frac{(x - 4)(x + 4)}{3x(x + 4)}$ is undefined when $x = 0$ or $x = -4$.

So, $f(x)$ is continuous for $x \neq 0$ and $x \neq -4$.

- (b) i. Note that $2e^x \rightarrow \infty$ as $x \rightarrow \infty$ and that $\frac{x^2 - 16}{3x^2 + 12x} \rightarrow \frac{1}{3}$ as $x \rightarrow \infty$ since the degree in the numerator and denominator are equal.

$$\text{So, } \lim_{x \rightarrow \infty} f(x) = \infty.$$

- ii. Note that $2e^x \rightarrow 0$ as $x \rightarrow -\infty$ and that $\frac{x^2 - 16}{3x^2 + 12x} \rightarrow \frac{1}{3}$ as $x \rightarrow -\infty$ since the degree in the numerator and denominator are equal.

$$\text{So, } \lim_{x \rightarrow -\infty} f(x) = \frac{1}{3}.$$

- (c) Again, note that e^x is continuous for all values.

Given that $\frac{x^2 - 16}{3x^2 + 12x} = \frac{(x - 4)(x + 4)}{3x(x + 4)} = \frac{x - 4}{3x}$, $f(x)$ has a vertical asymptote at $x = 0$, but not at $x = 4$.

The limits at infinity from part (b) give us information about the horizontal asymptotes.

So, $f(x)$ has the vertical asymptote $x = 0$ and the horizontal asymptote $y = \frac{1}{3}$.