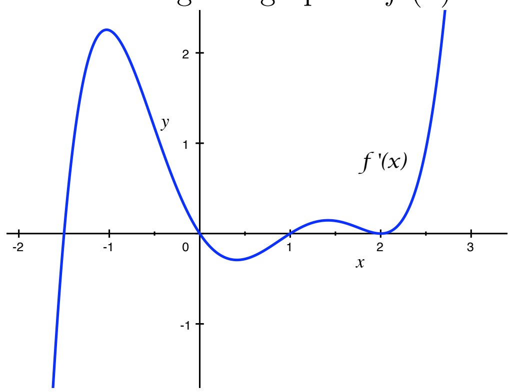
# Math 124 Worksheet #7 Solutions

1. The following is a graph of $f'(x)$.




(a) For what values of $x$ is $f$ increasing? Decreasing?

The function $f$ is increasing when $f'(x) > 0$. So, $f$ is increasing for $-1.5 < x < 0$, $1 < x < 2$, and $x > 2$.
The function $f$ is decreasing when $f'(x) < 0$. So, $f$ is decreasing for $x < -1.5$ and $0 < x < 1$.

(b) At what values of $x$ does $f$ have a local max? A local min?

Since $f'$ switches from positive to negative at $x = 0$, there is a local max at $x = 0$. Since $f'$ switches from negative to positive at $x = -1.5$ and $x = 1$, there are local mins at $x = -1.5$ and $x = 1$.

(c) For what values of $x$ is $f$ concave up? Concave down?

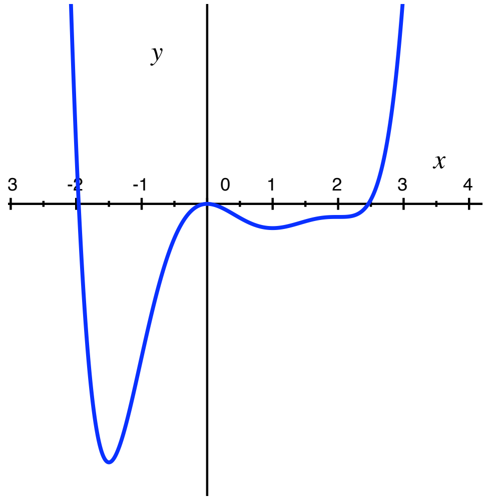
Note that when $f'$ is increasing, $f''(x) > 0$. So, when $f'$ is increasing, $f$ is concave up. $\Rightarrow$ $f$ is concave up for $x < -1$, $0.4 < x < 1.4$, and $x > 2$.

When $f'$ is decreasing, $f''(x) < 0$. So, when $f'$ is decreasing, $f$ is concave down. $\Rightarrow$ $f$ is concave down for $-1 < x < 0.4$ and $1.4 < x < 2$.

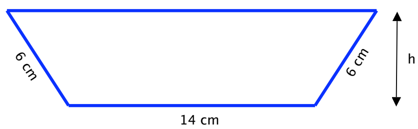
(d) What are the $x$-coordinates of the inflection points of $f$?

The inflection points of $f$ are the points at which $f$ switches concavity (or when $f''$ switches sign). Given the answer to part (c), the inflection points occur when $x = -1$, $x = 0.4$, $x = 1.4$, and $x = 2$.

(e) Assuming $f(0) = 0$, give a rough sketch of $f$.

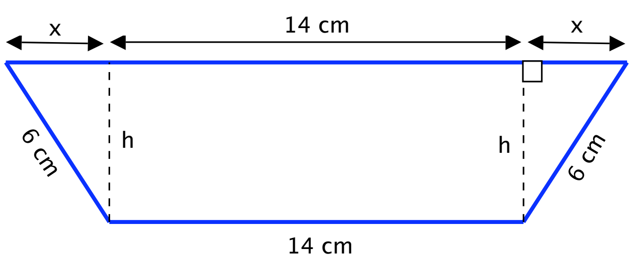



2. An isosceles trapezoid has a base of 14 cm and slant sides of 6 cm as shown in the figure below. What is the largest area of such a trapezoid?




There are many ways to solve this depending on the chosen variables. Here are a couple of ways:

- Let $x$ be the length as shown in the diagram below.




$\Rightarrow \quad h = \sqrt{36 - x^2}$
(Pythagorean theorem)

Note that the area of a trapezoid is given by $Area = \frac{1}{2}(B_1 + B_2)h$ where $B_1$ and $B_2$ are the lengths of the bases.
So, we have that $Area = A(x) = \frac{1}{2}(14 + 14 + 2x)\sqrt{36 - x^2} = (14 + x)\sqrt{36 - x^2}$.

Also note that we can consider $x$ such that $0 \leq x \leq 6$. (Taking $x$ negative will result in smaller trapezoids and $x$ cannot be longer than the sides of the trapezoid.)

Finding the critical numbers of $A(x)$:

$$A'(x) = 1 \cdot \sqrt{36 - x^2} + (14 + x)\frac{1}{2}(36 - x^2)^{-1/2} \cdot (-2x)$$
$$= \sqrt{36 - x^2} + \frac{-14x - x^2}{\sqrt{36 - x^2}}$$

This is undefined when $x = \pm 6$.

$$A'(x) = 0 \quad \Rightarrow \quad \sqrt{36 - x^2} + \frac{-14x - x^2}{\sqrt{36 - x^2}} = 0$$

$$\Rightarrow \quad 36 - x^2 - 14x - x^2 = 0 \quad \text{(Multiplying by } \sqrt{36 - x^2}\text{)}$$

$$-2x^2 - 14x + 36 = 0$$

$$x^2 + 7x - 18 = 0$$

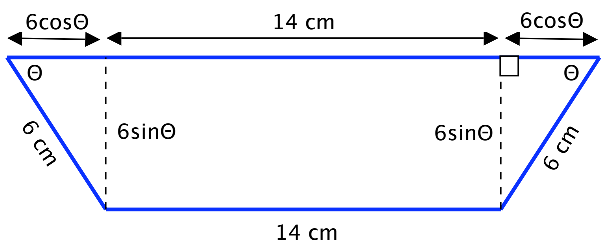
$$x = -9 \quad \text{or} \quad x = 2$$

Plugging in all the endpoints and critical numbers of interval into area function:

$$A(0) = 84 \qquad A(2) = 16\sqrt{32} \approx 90.51$$
$$A(6) = 0$$

So, the maximum area is approximately 90.52 $\text{cm}^2$.

- Let $\theta$ be the angle as shown in the diagram below.




Then the lengths of the sides of the triangles are $6\cos\theta$ and $6\sin\theta$ as shown in the diagram.

Note that the area of a trapezoid is given by $Area = \frac{1}{2}(B_1 + B_2)h$ where $B_1$ and $B_2$ are the lengths of the bases.
So, we have that $Area = A(\theta) = \frac{1}{2}(14 + 14 + 2(6\cos\theta))(6\sin\theta)$.
$= (14 + 6\cos\theta) \cdot 6\sin\theta$

Also note that we can consider $\theta$ such that $0 \leq \theta \leq \frac{\pi}{2}$ . (Taking larger values of $\theta$ will result in smaller trapezoids.)

Finding the critical numbers of $A(\theta)$:

$$\begin{aligned} A'(\theta) &= -6\sin\theta \cdot 6\sin\theta + (14 + 6\cos\theta) \cdot 6\cos\theta \\ &= -36\sin^2\theta + 84\cos\theta + 36\cos^2\theta \\ &= -36(1 - \cos^2\theta) + 84\cos\theta + 36\cos^2\theta \\ &= 72\cos^2\theta + 84\cos\theta - 36 \end{aligned}$$

$$A'(\theta) = 0 \quad \Rightarrow \quad 72\cos^2\theta + 84\cos\theta - 36 = 0$$

$$6\cos^2\theta + 7\cos\theta - 3 = 0$$

$$(3\cos\theta - 1)(2\cos\theta + 3) = 0 \quad \text{(You can also use quadratic formula to solve for } \cos\theta\text{.)}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{3} \quad \text{or} \quad \cos\theta = -\frac{3}{2}$$

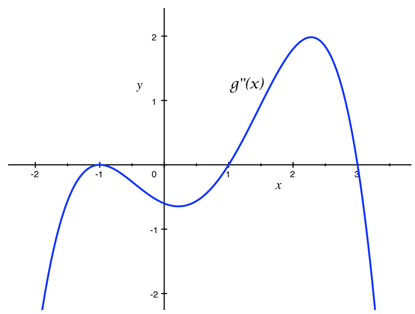
With our restrictions on $\theta$ ($0 \leq \theta \leq \frac{\pi}{2}$), we have the critical number $\theta = \arccos(\frac{1}{3}) \approx 1.231$.

Plugging in all the endpoints and critical numbers of interval into area function:

$$A(0) = 0 \qquad A(\arccos(\tfrac{1}{3})) = 64\sqrt{2} \approx 90.51$$
$$A(\tfrac{\pi}{2}) = 84$$

So, the maximum area is approximately 90.52 cm$^2$.

3. The following is a graph of $g''(x)$.




   (a) For what values of $x$ is $g$ concave up? Concave down?

   The function $g$ is concave up when $g''(x) > 0$. So, $g$ is concave up when $1 < x < 3$ or $(1, 3)$.
   The function $g$ is concave down when $g''(x) < 0$. So, $g$ is concave down when $x < -1$, $-1 < x < 1$, and $x > 3$ or on the intervals $(-\infty, -1)$, $(-1, 1)$, and $(3, \infty)$.

   (b) What are the $x$-coordinates of the inflection points of $g$?

   The function $g$ has an inflection point when it switches concavity which is where $g''(x)$ switches sign. Since $g''(x)$ switches from negative to positive at $x = 1$ and from positive to negative at $x = 3$, we have inflection points at $x = 1$ and $x = 3$.

4. A box with a square base and open top must have volume of 4000 in$^3$. Find the dimensions of the box that minimizes the amount of material used.

The volume of a box is $Volume = lwh$. Note that since the base of the box is square, the length and width of the box are the same. Let $x$ = length of box = width of box. Let $h$ = height of the box. Then the volume of the box is $x^2h$. Since the box must have volume 4000 in$^3$, we must have $x^2h = 4000 \quad \Rightarrow \quad h = \frac{4000}{x^2}$.

Since the box is has no top, the amount of material used to make the box will be the amount used to make the bottom in addition to the amount needed to make the 4 sides.

$$Material\ used = M(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{4000}{x^2} = x^2 + \frac{16000}{x}$$

Note that since $x$ is a length, we must have that $x > 0$ to have volume 4000 in$^3$.

Finding the critical numbers of $M(x)$:

$$M'(x) = 2x - \frac{16000}{x^2}$$

This is undefined for $x = 0$, but this is not in our domain.

$$M'(x) = 0 \quad \Rightarrow \quad 2x - \frac{16000}{x^2} = 0$$
$$2x^3 - 16000 = 0 \quad \text{(Multiplying both sides by } x^3\text{.)}$$
$$x^3 = 8000$$
$$x = 20$$

So, $x = 20$ is the critical number of $M(x)$.

Consider a sign chart for $M'(x)$:

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M'(x)    −    −    0    +    +
      |−−−−−−|−−−−−−−→
x     0          20
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So, $M(x)$ is decreasing for $0 < x < 20$ and increasing for $x > 20$. So, on our domain $x > 0$, we must have an absolute minimum at $x = 20$.

So, the dimensions of the box that minimize the amount of material used are $20'' \times 20'' \times 10''$ ($h = \frac{4000}{20^2} = 10$).

5. Find an equation for the slant asymptote of $h(x) = \frac{4x^4+x^3}{2x^3+2}$.

To find the slant asymptote, we need to do polynomial long division to simplify $h(x)$.

$$\begin{array}{rl} & 2x + \frac{1}{2} \\ 2x^3 + 2 \;) & 4x^4 + x^3 \\ & \underline{-(4x^4 + 4x)} \quad \text{(Mult. all terms of } 2x^3+2 \text{ by } 2x \text{ and subtract from } 4x^4+x^3\text{.)} \\ & x^3 - 4x \\ & \underline{-(x^3 + 1)} \quad \text{(Mult. } 2x^3+2 \text{ by } \tfrac{1}{2} \text{ and subtract from } x^3 - 4x\text{.)} \\ & -4x - 1 \end{array}$$

So, $h(x) = 2x + \frac{1}{2} + \frac{-4x-1}{2x^3+2}$. As $x \to \infty$, $\frac{-4x-1}{2x^3+2} \to 0$, so $h(x)$ approaches the line $2x + \frac{1}{2}$.

Thus, the slant asymptote of $h(x)$ is $y = 2x + \frac{1}{2}$.