Math 124 Worksheet #5 Solutions

1. Find all of the points of the graph $3x^2 + 4y^2 + 3xy = 26$ where the tangent line is horizontal. (Hint: After differentiating, plug in $\frac{dy}{dx} = 0$ to simplify your expression.)

   Differentiating both sides w.r.t. x: $\frac{d}{dx}(3x^2 + 4y^2 + 3xy) = \frac{d}{dx}(26)$

   $$6x + 8y\frac{dy}{dx} + 3 \cdot y + 3x \cdot \frac{dy}{dx} = 0 \quad \text{(Product Rule for } 3xy\text{)}$$

   Instead of solving for $\frac{dy}{dx}$, we just want to find values of x and y for which $\frac{dy}{dx} = 0$, so we can plug $\frac{dy}{dx} = 0$ into the equation.

   $$\Rightarrow \quad 6x + 8y(0) + 3y + 3x(0) = 0 \quad \Rightarrow \quad 6x + 3y = 0$$
   $$\Rightarrow \quad y = -2x$$

   So, if a point (x, y) is on the curve, it must satisfy the equation $3x^2 + 4y^2 + 3xy = 26$. If the slope of the curve at (x, y) is 0, then we must have that $y = -2x$. The values of x and y must satisfy both equations. To find the x and y values that satisfy these equations, it is easiest to plug $y = -2x$ into the first equation.

   $$\Rightarrow \quad 3x^2 + 4(-2x)^2 + 3x(-2x) = 26$$
   $$3x^2 + 16x^2 - 6x^2 = 26$$
   $$13x^2 = 26$$
   $$x^2 = 2$$
   $$x = \pm\sqrt{2}$$

   Since $y = -2x$, we have that when $x = \sqrt{2}$, $y = -2\sqrt{2}$ and when $x = -\sqrt{2}$, $y = 2\sqrt{2}$. So, the points at which the curve has horizontal tangents are $(\sqrt{2}, -2\sqrt{2})$ and $(-\sqrt{2}, 2\sqrt{2})$.

2. Find the 20th derivative of $\ln x$. (Note: An alternative notation for the 20th derivative of $\ln x$ is $D^{20} \ln x$.)

   Consider the first few derivatives of $f(x) = \ln x$.
   1st Derivative: $f'(x) = \frac{1}{x} = x^{-1}$
   2nd Derivative: $f''(x) = (-1)x^{-2}$
   3rd Derivative: $f'''(x) = (-1)(-2)x^{-3} = (-1)^2(1)(2)x^{-3}$
   4th Derivative: $f^{(4)}(x) = (-1)(-2)(-3)x^{-4} = (-1)^3(1)(2)(3)x^{-4}$

   In general: nth Derivative: $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$

   The 20th derivative is $-19!x^{-20} = -\dfrac{19!}{x^{20}}$.

3. $\frac{d}{dx}(\ln(xe^x) + 8) = ?$

   2 Options:

- $\frac{d}{dx}(\ln(xe^x)+8) = \frac{1}{xe^x}\cdot(1\cdot e^x + x\cdot e^x)+0$ (Product Rule on derivative of xe^x)
  $= \frac{1}{x}+1$

- Note that $\ln(xe^x) = \ln x + \ln e^x = \ln x + x$.
  So, $\frac{d}{dx}(\ln(xe^x)+8) = \frac{d}{dx}(\ln x + x + 8) = \frac{1}{x}+1$

4. If $f(t) = (\tan t)^{2t}$, what is the equation of the tangent line of f at the point $(\frac{\pi}{4}, 1)$?

   If $y = (\tan t)^{2t}$, then $\ln y = 2t\ \ln(\tan t)$. Differentiating both sides:

$$\frac{1}{y}\frac{dy}{dt} = 2\ \ln(\tan t) + 2t\cdot \frac{1}{\tan t}\cdot \sec^2 x$$
$$\Rightarrow \frac{dy}{dt} = (\tan t)^{2t}(2\ \ln(\tan t) + 2t\cdot \frac{1}{\tan t}\cdot \sec^2 x)$$

   Plugging in $x = \frac{\pi}{4}$, we get that $\frac{dy}{dt} = \pi$.

   So, an equation of the tangent line is $y - 1 = \pi(x - \frac{\pi}{4})$.

5. A golf cart has the position $s = f(t) = 2\text{arctan}\ t$ (in miles) at a time t (in hour). When is the cart speeding up? When is the cart slowing down?

   The velocity of the cart (in miles/hour) is given by the derivative of $f(t)$.

$$\Rightarrow \quad v = f'(t) = \frac{2}{1+t^2}$$

   Note that $t^2 \geq 0$ for any value of t, so the velocity is always positive.

   The acceleration of the cart (in miles/hour^2) is given by the 2nd derivative of $f(t)$.

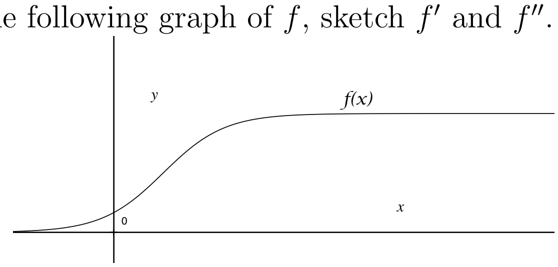
$$\Rightarrow \quad a = f''(t) = \frac{-4t}{(1+t^2)^2}$$

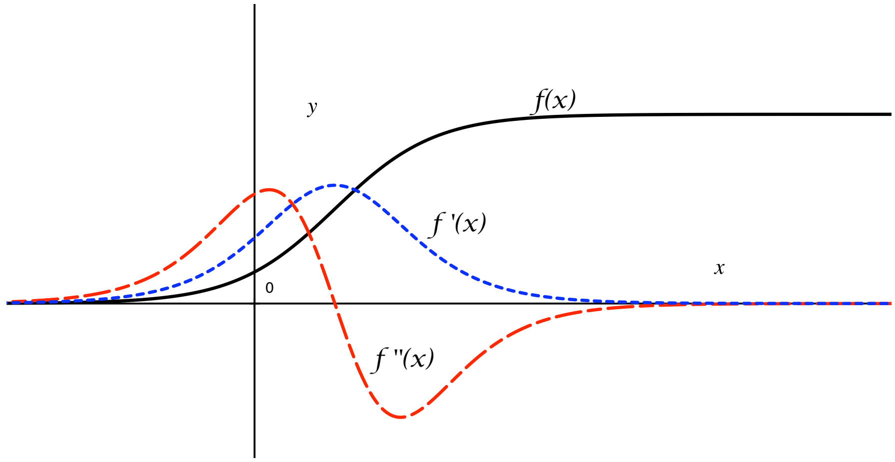
   Note that since we only consider positive values for time ($t \geq 0$), acceleration is always negative.

   The cart will be speeding up at a given time if the velocity and acceleration match in sign, i.e., the velocity and acceleration are simultaneously positive or the velocity and acceleration are simultaneously negative.

   Since the velocity and acceleration are always opposite in sign, the cart is slowing down for all time values.

6. For the following graph of f, sketch f' and f''.





y
f(x)
f '(x)
x
0
f ''(x)