Math 124 Worksheet #4 Solutions

1. $f(\theta) = \sqrt{3}\cos\theta - \sin\theta$

   (a) Find the roots/zeros of the function f.

   We want to find the θ-values for which $f(\theta) = 0$.

   $$\begin{aligned} \Rightarrow \quad \sqrt{3}\cos\theta - \sin\theta &= 0 \\ \sqrt{3}\cos\theta &= \sin\theta \\ \sqrt{3} &= \tan\theta \quad \text{(Div. by } \cos\theta\text{)} \end{aligned}$$

   Note: $\arctan(\sqrt{3}) = \frac{\pi}{3}$, so $\tan\theta = \sqrt{3}$ when $\theta = \frac{\pi}{3} + k\pi$ for an integer k

   Thus, the roots of f are $\theta = \frac{\pi}{3} + k\pi$ for an integer k.

   (b) For what values of θ does f have a horizontal tangent?
   The function f will have a horizontal tangent when $f'(\theta) = 0$.

   $$\begin{aligned} \Rightarrow \quad -\sqrt{3}\sin\theta - \cos\theta &= 0 \\ -\sqrt{3}\sin\theta &= \cos\theta \\ -\sqrt{3}\tan\theta &= 1 \\ \tan\theta &= -\frac{1}{\sqrt{3}} \end{aligned}$$

   Note: $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$, so $\tan\theta = -\frac{1}{\sqrt{3}}$ when $\theta = -\frac{\pi}{6} + k\pi$ for an integer k

   Thus, f has horizontal tangents at the values $\theta = -\frac{\pi}{6} + k\pi$ for an integer k.

   (c) What are the maximum and minimum values of the function? (Consider the graph of the function.)

   Considering the graph of the function, we can see that the maximum and minimum values of f will occur when f has a horizontal tangent. So, f has a maximum or minimum when $\theta = -\frac{\pi}{6} + k\pi$ for an integer k. Plugging the values of θ into the original function f will yield either 2 or -2, so the maximum value of f is 2 and the minimum value is -2.

2. Evaluate the following.

   (a) $\frac{d}{dx}\left(\frac{e^x \tan x}{\cos x}\right)$

   2 Options:

   - $$\begin{aligned} \frac{d}{dx}\left(\frac{e^x \tan x}{\cos x}\right) &= \frac{\cos x(e^x\tan x + e^x\sec^2 x) - e^x\tan x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos x(e^x\tan x + e^x\sec^2 x) + e^x\tan x \sin x}{\cos^2 x} \end{aligned}$$

(Quotient Rule with the Product Rule for the derivative of the numerator)

- $\frac{d}{dx}\left(\frac{e^x \tan\ x}{\cos\ x}\right) = \frac{d}{dx}(e^x \tan\ x\ \sec\ x)$
$= e^x \frac{d}{dx}(\tan\ x\ \sec\ x) + e^x \tan\ x\ \sec\ x$ (Product Rule)
$= e^x(\sec^2 x\ \sec\ x + \tan\ x\ \sec\ x\ \tan\ x) + e^x \tan\ x\ \sec\ x$ (Again)
$= e^x\ \sec\ x(\sec^2 x + \tan^2 x + \tan\ x)$

(b) If $g(t) = (2 + 3t + 5t^3)^7 \cos\ t$, what is $g'(0)$?

Using the product rule and then the chain rule, we have

$$g'(t) = \frac{d}{dt}((2 + 3t + 5t^3)^7) \cdot \cos\ t + (2 + 3t + 5t^3)^7 \cdot \frac{d}{dt}(\cos\ t)$$

$$= 7(2 + 3t + 5t^3)^6(3 + 15t^2)\cos\ t - (2 + 3t + 5t^3)^7 \sin\ t$$

So, $g'(0) = 7(2+3(0)+5(0)^3)^6(3+15(0)^2)\cos(0) - (2+3(0)+5(0)^3)^7 \sin(0) = 1344$.

(c) $\frac{d}{dr}(5^{\csc(\pi r)} + 8e^{\sqrt{r}})$

Using the sum rule and then the chain rule (multiple times), we have

$$\frac{d}{dr}(5^{\csc(\pi r)} + 8e^{\sqrt{r}}) = \frac{d}{dr}(5^{\csc(\pi r)}) + \frac{d}{dr}(8e^{\sqrt{r}})$$
$$= 5^{\csc(\pi r)} \ln\ 5 \cdot (-\csc(\pi r)\cot(\pi r)) \cdot \pi + 8e^{\sqrt{r}} \cdot \tfrac{1}{2} r^{-1/2}$$
$$= -5^{\csc(\pi r)} \ln\ 5 \cdot \csc(\pi r)\cot(\pi r) \cdot \pi + 4e^{\sqrt{r}} \cdot r^{-1/2}$$

3. Find an equation of the tangent line to the curve $x^2 - \sin(xy) + y^3 = 1$ at the point $(1, 0)$.

Differentiating both sides with respect to x:

$$\frac{d}{dx}(x^2 - \sin(xy) + y^3) = \frac{d}{dx}(1)$$
$$2x - \cos(xy)(1 \cdot y + x\tfrac{dy}{dx}) + 3y^2 \tfrac{dy}{dx} = 0 \quad \text{(Chain Rule \& Product Rule)}$$

Plugging in $x = 1$ and $y = 0$:

$$2(1) - \cos(1 \cdot 0)(1 \cdot 0 + 1 \cdot \tfrac{dy}{dx}) + 3(0)^2 \tfrac{dy}{dx} = 0$$
$$2 - 1 \cdot (0 + \tfrac{dy}{dx}) = 0$$
$$2 - \tfrac{dy}{dx} = 0$$

$$\text{So, } \frac{dy}{dx} = 2.$$

Thus, an equation of the tangent line is $y = 2(x - 1)$.