Math 124 Worksheet #2 Solutions

1. For what values is $\ln(\tan^2 x) + 2x^5$ continuous?

   Note that $2x^5$ is continuous for all real numbers (domain is all real numbers).

   Also note that $\tan^2 x = (\tan\, x)^2$. So $\tan^2 x$ is defined and continuous when $\tan\, x$ is defined and continuous. The domain for $\tan\, x$ is all real numbers except $x = \frac{\pi}{2} + k\pi$, for an integer k.

   The function $\ln(\tan^2 x)$ will be continuous whenever it is defined. The domain of natural log is all positive numbers. So, for the composition $\ln(\tan^2 x)$ to be defined, we must have $\tan^2 x > 0$. Of particular concern are the values of x for which $\tan^2 x = 0$, which are the values $x = k\pi$ for an integer k.

   So, $\ln(\tan^2 x)$ is defined for all real numbers except $x = \frac{\pi}{2} + k\pi$ (values for which $\tan^2 x$ is undefined) and $x = k\pi$ (values for which $\tan^2 x = 0$) for an integer k.

   You can write this in one statement as $\ln(\tan^2 x)$ is defined for all real numbers except $x = \frac{\pi}{2} + \frac{k\pi}{2}$ for an integer k.

2. For what value of c is the function below continuous at $x = 0$?

   $$f(x) = \{\, c\, e^{x^2 - x}, \quad \text{if} x \leq 0$$
   $$\{2x^2 + 1 + 2c, \ \text{if} x > 0$$

   For this function to be continuous at $x = 0$, we must have that the limit as x approaches 0 exists. So, the left and right-hand limits must equal.

   $$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^+} f(x) \\ \lim_{x \to 0^-} ce^{x^2 - x} &= \lim_{x \to 0^+} 2x^2 + 1 + 2c \\ ce^{0^2 - 0} &= 2(0)^2 + 1 + 2c \\ c &= 1 + 2c \\ c &= -1 \end{aligned}$$

   So, c must equal -1 for the function $f(x)$ to be continuous.

3. Evaluate the following limits.

   (a) $\lim_{x \to -\infty} \frac{x+1}{3x^5 - 2x + 5}$

$$\lim_{x\to-\infty} \frac{x+1}{3x^5-2x+5} = \lim_{x\to-\infty} \frac{x+1}{3x^5-2x+5} \cdot \frac{\frac{1}{x^5}}{\frac{1}{x^5}} = \lim_{x\to-\infty} \frac{\frac{1}{x^4}+\frac{1}{x^5}}{3-\frac{2}{x^4}+\frac{5}{x^5}} = \frac{0}{3} = 0$$

(b) $\lim_{x\to\infty} \frac{\sqrt{9x^2+x+1}}{2x-5}$

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+x+1}}{2x-5} = \lim_{x\to\infty} \frac{\sqrt{9x^2+x+1}}{2x-5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}(9x^2+x+1)}}{2-\frac{5}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}}{2-\frac{5}{x}} = \frac{\sqrt{9}}{2} = \frac{3}{2}$$

4. Estimate $\lim_{x\to\infty}(1+\frac{1}{x})^x$ with a table of values (to 4 decimal places). Does this value look familiar?

Table of Values:

x	$f(x)$
100	≈ 2.7048138
1000	≈ 2.7169239
10000	≈ 2.7181459
100000	≈ 2.7182682

As $x \to \infty$, the function $(1+\frac{1}{x})^x$ approaches the value $e \approx 2.718281828$.

5. Find the horizontal and vertical asymptotes of the curve $y = \frac{3x^5-1}{x^2+5x+6}$.

We can find the vertical asymptotes by factoring the denominator and finding its roots. $x^2+5x+6 = (x+3)(x+2) \Rightarrow$ The denominator is 0 when $x=-2$ or $x=-3$. Since $x=-2$ and $x=-3$ are not roots of the polynomial in the numerator, the lines $x=-2$ and $x=-3$ are vertical asymptotes of the rational function.

To find horizontal asymptotes, we must look at the end behavior of the function, i.e., the limits at infinity.

$$\lim_{x\to\infty} \frac{3x^5-1}{x^2+5x+6} = \lim_{x\to\infty} \frac{3x^3-\frac{1}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}} = \lim_{x\to\infty} 3x^3 = \infty$$

$$\lim_{x\to-\infty} \frac{3x^5-1}{x^2+5x+6} = \lim_{x\to-\infty} \frac{3x^3-\frac{1}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}} = \lim_{x\to-\infty} 3x^3 = -\infty$$

Since the limits at infinity are infinite (do not exist), this function has no horizontal asymptotes.