Math 124
Final Exam Solutions

1. (12 pts.) Evaluate the following limits. Justify your answers.

   (a) (4 pts.) $\lim_{x\to 2} \frac{\frac{2}{x^2}-\frac{1}{2}}{x-2}$

   Here are two ways to evaluate the limit:

   - $$\begin{aligned}\lim_{x\to 2} \frac{\frac{2}{x^2}-\frac{1}{2}}{x-2} &= \lim_{x\to 2} \frac{\frac{4-x^2}{2x^2}}{x-2} \qquad \text{Common denominator of } 2x^2\\ &= \lim_{x\to 2} \frac{4-x^2}{2x^2(x-2)}\\ &= \lim_{x\to 2} \frac{(2+x)(2-x)}{2x^2(x-2)}\\ &= \lim_{x\to 2} \frac{-(2+x)}{2x^2}\\ &= \frac{-(2+2)}{2(2)^2} = -\frac{1}{2}\end{aligned}$$

   - Note that if $f(x) = \frac{2}{x^2}$ and $a = 2$, then the limit is the derivative of f at $a = 2$.
     $$\lim_{x\to 2} \frac{\frac{2}{x^2}-\frac{1}{2}}{x-2} = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = f'(2).$$
     Note that since $f'(x) = \frac{-4}{x^3}$, $f'(2) = -\frac{1}{2}$. So, the limit is $-\frac{1}{2}$.

   (b) (4 pts.) $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{\sqrt{9x^2-5}}$

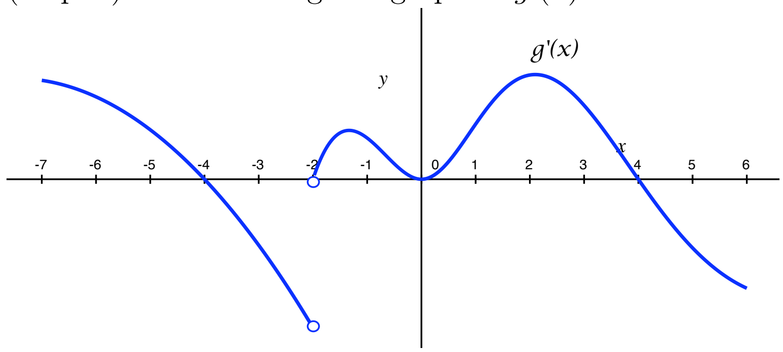
   $$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{\sqrt{9x^2-5}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}(\frac{1}{x})}{\sqrt{9x^2-5}(\frac{1}{x})} = \lim_{x\to\infty} \frac{\sqrt{(x^2+1)(\frac{1}{x^2})}}{\sqrt{(9x^2-5)(\frac{1}{x^2})}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{\sqrt{9-\frac{5}{x^2}}} = \frac{\sqrt{1}}{\sqrt{9}} = \frac{1}{3}$$

   (c) (4 pts.) $\lim_{x\to 2^-} \frac{|x-2|}{x^2-2x}$

   Note that
   $$|x-2| = \begin{cases} x-2 & \text{for } x \geq 2 \\ -(x-2) & \text{for } x < 2 \end{cases}$$

   So $\lim_{x\to 2^-} \frac{|x-2|}{x^2-2x} = \lim_{x\to 2^-} \frac{-(x-2)}{x^2-2x} = \lim_{x\to 2^-} \frac{-(x-2)}{x(x-2)} = \lim_{x\to 2^-} \frac{-1}{x} = -\frac{1}{2}$.

2. (12 pts.) The following is a graph of $g'(x)$.



(a) (3 pts.) On what intervals is the function $g(x)$ increasing? On what intervals is $g(x)$ decreasing?

The function $g(x)$ is increasing when $g'(x) > 0$. So, g is increasing on the intervals $(-7, -4)$, $(-2, 0)$, and $(0, 4)$.
The function $g(x)$ is decreasing when $g'(x) < 0$. So, g is decreasing on the intervals $(-4, -2)$ and $(4, 6)$.

(b) (3 pts.) Assuming the function g is defined for all values in $[-7, 6]$, does g have local extreme values? If so, at what x-values does g have a local maximum? At what x-values does g have a local minimum?

Note that g has the critical numbers $x = -4$, $x = -2$, $x = 0$, and $x = 4$ (values of x at which g' is 0 or undefined). There are local maxima at $x = -4$ and $x = 4$ since g' switches from positive to negative and there is a local minimum at $x = -2$ since g' switches from negative to positive. (The function g must have a corner at $x = -2$.)

(c) (3 pts.) For what x-values is the function g concave up? For what x-values is g concave down?

Note that $g(x)$ is concave up if $g''(x) > 0$ and $g''(x) > 0$ if $g'(x)$ is increasing. Since $g'(x)$ is increasing on the approximate intervals $(-2, -1.4)$ and $(0, 2)$, the function g is concave up on those intervals. Since $g'(x)$ is decreasing on the approximate intervals $(-7, -2)$, $(-1.4, 0)$, and $(2, 6)$, the function g is concave down on those intervals.

(d) (3 pts.) What are the x-coordinates of the inflection points of g?

Since the function g switches concavity at approximately $x = -2$, $x = -1.4$, $x = 0$, and $x = 2$, the function has inflection points at those values of x.

3. (16 pts.) Evaluate the following. You do not need to simplify your answer.

(a) (5 pts.) $\frac{d}{dt}[3t + 2\sqrt[3]{\arctan(3t)} - e^{\pi+1}]$

$$\frac{d}{dt}[3t + 2\sqrt[3]{\arctan(3t)} - e^{\pi+1}] = 3 + 2 \cdot \tfrac{1}{3}(\arctan(3t))^{-2/3} \cdot \tfrac{1}{1+(3t)^2} \cdot 3 - 0 \quad \text{(Sum and chain rule)}$$

(b) (5 pts.) $\frac{d}{dx}\left[\frac{\sin^2(4x)}{\cos(x^3)}\right] = ?$

$$\frac{d}{dx}\left[\frac{\sin^2(4x)}{\cos(x^3)}\right] = \frac{\cos(x^3) \cdot 2\sin(4x) \cdot \cos(4x) \cdot 4 - \sin^2(4x) \cdot (-\sin(x^3)) \cdot 3x^2}{\cos^2(x^3)}$$

(c) (6 pts.) If $g(t) = (1-t)^{e^t}$, what is $g'(t)$?

We will have to use logarithmic differentiation since we have a variable expression in the base and exponent.

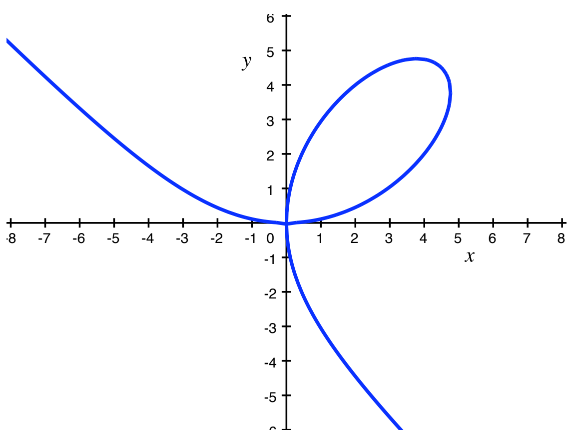
Let $y = (1-t)^{e^t}$. Then $\ln y = \ln((1-t)^{e^t}) = e^t \cdot \ln(1-t)$.

Taking the derivative of both sides with respect to t:

$$\frac{1}{y} \cdot \frac{dy}{dt} = e^t \cdot \ln(1-t) + e^t \cdot \tfrac{1}{1-t} \cdot (-1) = e^t \cdot \ln(1-t) - \tfrac{e^t}{1-t}$$

$$\text{So, } \frac{dy}{dt} = g'(t) = (1-t)^{e^t}\left(e^t \cdot \ln(1-t) - \tfrac{e^t}{1-t}\right).$$

4. (14 pts.) The curve shown below is $x^3 + y^3 = 9xy$.



(a) (2 pts.) Verify that the point $(2,4)$ is on the curve.

Left-side of equation: $2^3 + 4^3 = 72$
Right-side of equation: $9(2)(4) = 72$

So, the point $(2,4)$ is on the curve since $x = 2$ and $y = 4$ satisfies the equation.

(b) (7 pts.) Find the equation of the tangent line of the curve at the point $(2,4)$.

We need the slope of the curve at $(2,4)$, so we must find $\frac{dy}{dx}$ when $x = 2$ and $y = 4$.

To find $\frac{dy}{dx}$, we must take the derivative of both sides of the equation with respect to x.

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[9xy]$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 9y + 9x \cdot \frac{dy}{dx}$$

Evaluating the equation when $x = 2$, $y = 4$:

$$3(2)^2 + 3(4)^2\frac{dy}{dx} = 9(4) + 9(2)\frac{dy}{dx}$$

$$12 + 48\frac{dy}{dx} = 36 + 18\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{24}{30} = \frac{4}{5}$$

So the equation of the tangent line at $(2, 4)$ is $\quad y - 4 = \frac{4}{5}(x - 2)$.

(c) (5 pts.) Let $P = (x_1, 4.1)$ be a point on the curve near the point $(2, 4)$. Using part (b) and linear approximation, estimate x_1.

If P is a point near the point $(2, 4)$ on the curve, then x_1 can be approximated by using the tangent line from part (b) and finding the point on the line with y-coordinate equal to 4.1.

Plugging $y = 4.1$ into the equation of the tangent line we get

$$4.1 - 4 = \frac{4}{5}(x - 2) \quad \Rightarrow \quad x = 2.125$$

So, the point $(2.125, 4.1)$ in on the tangent line $\quad \Rightarrow \quad x_1 \approx 2.125$

5. (24 pts.) Let $f(x) = e^{2x}\sin(2x)$ on the domain $[-1, 2]$.

(a) (6 pts.) What are the critical numbers of f in the domain?

Finding the first derivative:

$$\begin{aligned} f'(x) &= e^{2x} \cdot 2 \cdot \sin(2x) + e^{2x} \cdot \cos(2x) \cdot 2 \quad \text{(Product Rule)} \\ &= 2e^{2x}(\sin(2x) + \cos(2x)) \end{aligned}$$

Note that f' is defined for all values of x, so the critical numbers of f will be x-values for which the derivative is equal to zero. Also note that $2e^{2x}$ is always positive and never zero, so we must find values for which $\quad \sin(2x) + \cos(2x) = 0$.

$$\begin{aligned} \sin(2x) &= -\cos(2x) \\ \tan(2x) &= -1 \quad \text{(Dividing both sides by } \cos(2x)) \\ \Rightarrow \quad 2x &= -\tfrac{\pi}{4} + k\pi \quad \text{for an integer } k \\ \Rightarrow \quad x &= -\tfrac{\pi}{8} + k\tfrac{\pi}{2} \quad \text{for an integer } k \end{aligned}$$

So, the critical numbers in the domain $[-1, 2]$ are $x = -\frac{\pi}{8}$ and $x = -\frac{\pi}{8} + \frac{\pi}{2} = \frac{3\pi}{8}$.

(b) (4 pts.) For what intervals in the domain is f increasing? For what intervals is f decreasing?

Consider a sign chart for f'.

f'	$- \; -$	0	$+ \; + \; + \; +$	0	$- \; - \; -$	
-1		$-\frac{\pi}{8}$		$\frac{3\pi}{8}$		2

So, f is decreasing on the intervals $(-1, -\frac{\pi}{8})$ and $(\frac{3\pi}{8}, 2)$ and increasing on the interval $(-\frac{\pi}{8}, \frac{3\pi}{8})$.

(c) (5 pts.) For what intervals in the domain is f concave up? For what intervals is f concave down?

Finding the second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}[2e^{2x}(\sin(2x)+\cos(2x))] \\ &= 2e^{2x}\cdot 2\cdot[\sin(2x)+\cos(2x)] + 2e^{2x}[\cos(2x)\cdot 2 - \sin(2x)\cdot 2] \\ &= 4e^{2x}[\sin(2x)+\cos(2x)+\cos(2x)-\sin(2x)] \\ &= 8e^{2x}\cos(2x) \end{aligned}$$

Note that f'' is equal to zero when $\cos(2x)=0 \quad \Rightarrow \quad 2x = \frac{\pi}{2}+k\pi$ for an integer k.

$\Rightarrow \quad x = \frac{\pi}{4} + k\frac{\pi}{2}$

So, in the interval $[-1,2]$, $f''(x)=0$ for $x=\pm\frac{\pi}{4}$.

Consider a sign chart for $f''(x)$.

```
f''      −   −   −      0      +   +   +   +      0      −   −   −
   |−−−−−−−−−−−−−−−−−−−|−−−−−−−−−−−−−−−−−−−−−−−−|−−−−−−−−−−−−−−−−−−|
  −1                 −π/4                      π/4                 2
```

So, f is concave up on the interval $(-\frac{\pi}{4},\frac{\pi}{4})$ and concave down on the intervals $(-1,-\frac{\pi}{4})$ and $(\frac{\pi}{4},2)$.

(d) (5 pts.) Find the local maximum and minimum values of f.

We have the critical numbers $x=-\frac{\pi}{8}$ and $x=\frac{3\pi}{8}$. You can use either the 1st or 2nd derivative test to classify these values.

- 1st Derivative Test (see part (b)):
  Since f' switches from negative to positive at $x=-\frac{\pi}{8}$, f has a local minimum at that value.
  Since f' switches from positive to negative at $x=\frac{3\pi}{8}$, f has a local maximum at that value.
- 2nd Derivative Test (see part (c)):
  Since f'' is positive at $x=-\frac{\pi}{8}$, f has a local minimum at that value.
  Since f'' is negative at $x=\frac{3\pi}{8}$, f has a local maximum at that value.

So, we have the local maximum value $f(\frac{3\pi}{8}) = e^{3\pi/4}\sin(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}e^{3\pi/4} \approx 7.4605$ and the local minimum value $f(-\frac{\pi}{8}) = e^{-\pi/4}\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}e^{-\pi/4} \approx -0.3224$.

(e) (4 pts.) What are the absolute maximum and minimum values of f?

We have already evaluated f at the critical numbers in the domain, so we only need to evaluate the function at the endpoints of the interval and compare all the values to find the extreme values.

$$f(-1) = e^{-2}\sin(-2) \approx -0.1231 \qquad f(2) = e^{4}\sin(4) \approx -41.32$$

So, the absolute maximum value is approximately 7.4605 (at $x = \frac{3\pi}{8}$) and the absolute minimum value is approximately -41.32 (at $x = 2$).

6. (13 pts.) The product of 2 positive numbers is 243. How small can the sum of one of the numbers plus the cube of the other number be? Show your work and justify that your solution is an absolute minimum.

Let x = 1st number $\quad\Rightarrow\quad xy = 243$
y = 2nd number

We want to minimize the following sum: $\quad Sum = x + y^3$
Since $x = \frac{243}{y}$, we have that $Sum = S(y) = \frac{243}{y} + y^3$

Note that we must have that $y > 0$ since both x and y must be positive numbers.

Finding critical numbers:
$S'(y) = -\frac{243}{y^2} + 3y^2$ (This is undefined for $y = 0$, but that is not in our domain.)

If $-\frac{243}{y^2} + 3y^2 = 0$, then $-243 + 3y^4 = 0$. (Mult. both sides by y^2)
$\Rightarrow\ y^4 = 81$
$\Rightarrow\ y = 3$

Consider a sign chart for $S'(y)$:

```
S'        −   −   −   −     0        +   +   +   +   +
    |-------------------------|----------------------------→
    0                         3
```

Since the function $S(y)$ is decreasing from $y = 0$ to $y = 3$ and then increasing for all values $y > 3$, we must have an absolute minimum at $y = 3$. So, the smallest possible sum is

$$S(3) = \frac{243}{3} + (3)^3 = 81 + 27 = 108.$$

7. (9 pts.)

$$h(t) = \begin{cases} t^2 - t - 2c & \text{for } t \geq -1 \\ -(t+2)^3 + c & \text{for } t < -1 \end{cases}$$

(a) (4 pts.) For what value of c is the function h continuous on $(-\infty, \infty)$?

Note that each part of the piecewise-defined function h is a polynomial, so h is continuous for all values less than -1 and greater than -1. To be continuous at $t = -1$, we must have that $\lim_{t \to -1} h(t) = h(-1)$.

In order for the limit at $t \to -1$ to exist, we must have that the left and right-hand limits exist and are equivalent. So, $\lim_{t \to -1^-} h(t) = \lim_{t \to -1^+} h(t)$.

$$\lim_{t \to -1^-} -(t+2)^3 + c = \lim_{t \to -1^+} t^2 - t - 2c$$

$$\begin{aligned} -(-1+2)^3 + c &= (-1)^2 + 1 - 2c \quad \text{(Evaluating limit of polynomials)} \\ -1 + c &= 2 - 2c \\ c &= 1 \end{aligned}$$

So, in order for the functions to match at $t = -1$, we must have that $c = 1$.

Given $c = 1$, we have that $h(-1) = 0 = \lim_{t \to -1} h(t)$. So, h is continuous if $c = 1$.

(b) (5 pts.) Given the value of c from part (a), is h differentiable at $t = -1$? Justify your answer.

In order for h to be differentiable at $t = -1$, we must have that the derivatives exists at $t = -1$, that is, the derivative from the left and the right are equal.

Note that the derivative of $h(t)$ for $t < -1$ is $-3(t+2)^2$ so from the left
$\lim_{t \to -1^-} \frac{h(t)-h(-1)}{t+1} = -3(-1+2)^2 = -3$.

Note that the derivative of $h(t)$ for $t > -1$ is $2t - 1$ so from the right
$\lim_{t \to -1^+} \frac{h(t)-h(-1)}{t+1} = 2(-1) - 1 = -3$.

Since the left and right-hand limits are equal (derivatives match from left and right at $t = -1$), the function h is differentiable at $t = -1$.