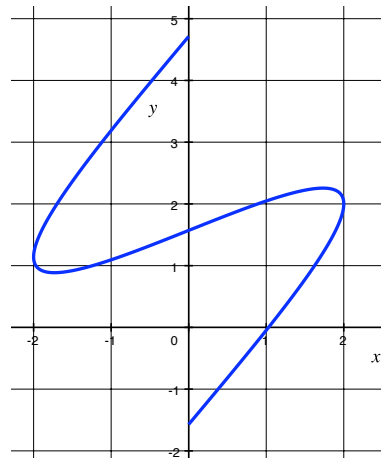


Math 124
Exam 3 Solutions

1. (20 pts.) The curve below is described by the equations $x(t) = 2\cos(t)$ for $-\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$.
 $y(t) = t + 2\cos(t)$

- (a) (10 pts.) Find an equation of the tangent line to the curve at the point at which $t = \pi$.



To write an equation of the tangent, we need the point at which $t = \pi$ and the slope of the curve at $t = \pi$.

The point at which $t = \pi$ is $(x(\pi), y(\pi)) = (2\cos(\pi), \pi + 2\cos(\pi)) = (-2, \pi - 2)$.

The slope of the tangent line is given by $\frac{dy}{dx}$ at $t = \pi$.

$$\begin{aligned} \text{Note that } x'(t) = -2\sin(t) &\Rightarrow x'(\pi) = -2\sin(\pi) = 0 \\ \text{and } y'(t) = 1 - 2\sin(t) &\Rightarrow y'(\pi) = 1 - 2\sin(\pi) = 1 \end{aligned}$$

So, $\frac{dy}{dx}\bigg|_{t=\pi} = \frac{y'(\pi)}{x'(\pi)}$, however since $x'(\pi) = 0$, the tangent line is vertical (undefined slope).

Equation of tangent line: $x = -2$

- (b) (10 pts.) Find the **values of t** at which the tangent line of the curve is horizontal.

The tangent line will be horizontal when $y'(t) = 0$.

Finding values of t at which $y'(t) = 0$:

$$1 - 2\sin(t) = 0 \Rightarrow \sin(t) = \frac{1}{2}$$

$$\Rightarrow t = \frac{\pi}{6} \quad \text{or} \quad t = \frac{5\pi}{6}$$

2. (20 pts.) $f(x) = 3x^{2/3}(x + 15)$

- (a) (12 pts.) Find the critical numbers of f on the interval $[-15, 1]$.

Differentiating f using the product rule:

$$f'(x) = 3\left(\frac{2}{3}\right)x^{-1/3}(x + 15) + 3x^{2/3}(1)$$

$$\begin{aligned}
&= 2x^{-1/3}(x + 15) + 3x^{2/3} \\
&= 2x^{2/3} + 30x^{-1/3} + 3x^{2/3} \\
&= \frac{30}{x^{1/3}} + 5x^{2/3}
\end{aligned}$$

Note that $f'(x)$ is undefined when $x = 0$, so $x = 0$ is a critical number.

Finding x such that $f'(x) = 0$:

$$\begin{aligned}
\frac{30}{x^{1/3}} + 5x^{2/3} &= 0 \\
30 + 5x &= 0 \quad (\text{Mult. both sides by } x^{1/3}) \\
\Rightarrow x &= -6
\end{aligned}$$

The function f has the critical numbers $x = 0$ and $x = -6$ on the interval $[-15, 1]$.

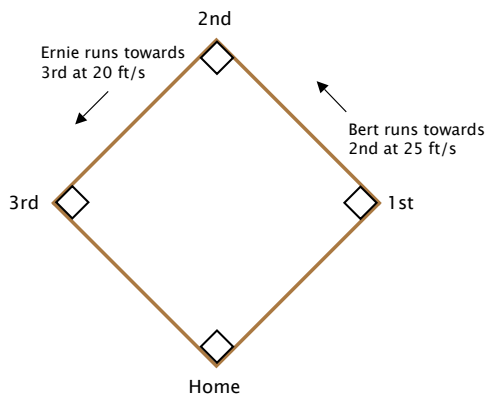
- (b) (8 pts.) Find the absolute maximum and minimum **values** of f on the interval $[-15, 1]$.

Evaluating f at the endpoints and critical points:

$$\begin{array}{ll}
f(-15) = 0 & f(0) = 0 \\
f(1) = 48 & f(-6) \approx 89.1520
\end{array}$$

The absolute maximum value of f is approximately 89.1520 and it occurs at $x = -6$.
The absolute minimum value of f is 0 and it occurs at $x = 0$ and $x = -15$.

3. (25 pts.) Bert is on first base and Ernie is on second base during a baseball game. When the batter hits the ball, Bert and Ernie simultaneously start running towards the next bases. Bert runs at a speed of 25 feet/second and Ernie runs at a speed of 20 feet/second. How fast is the distance between them changing when Bert has run 50 feet and Ernie has run 40 feet? Is the distance between them increasing or decreasing? (Recall: The sides of a baseball diamond are 90 feet in length.)

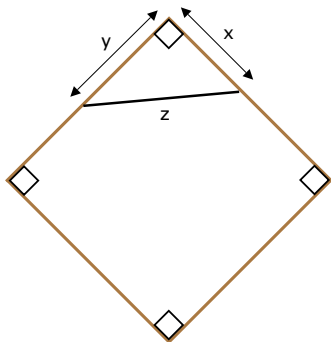


x = distance between Bert and 2nd base
 y = distance between Ernie and 2nd
 z = distance between Bert and Ernie

$$\Rightarrow \frac{dx}{dt} = -25 \text{ ft/sec} \quad \& \quad \frac{dy}{dt} = 20 \text{ ft/sec}$$

Unknown: $\frac{dz}{dt}$ when $x = 90 - 50 = 40$
and $y = 40$

Given our assignment of variables, we have the following diagram and equation to solve:



Equation: $x^2 + y^2 = z^2$

Differentiating with respect to t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

$$\Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = z \frac{dz}{dt}$$

Note that if $x = 40$ and $y = 40$, then $z = \sqrt{40^2 + 40^2} = 40\sqrt{2} \approx 56.5685$ feet.

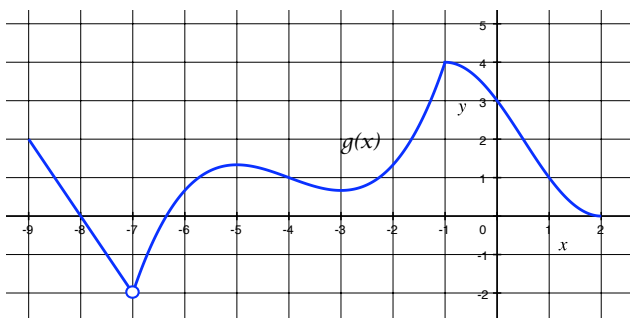
Plugging in values at the time at which $x = 40$ and $y = 40$:

$$40(-25) + 40(20) = 40\sqrt{2} \frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = -\frac{5}{\sqrt{2}} \text{ ft/sec}$$

So, the distance between Bert and Ernie is decreasing at a rate of $\frac{5}{\sqrt{2}}$ feet/sec.

4. (15 pts.) The graph of $g(x)$ on the interval $[-9, 2]$ is given below.



- (a) (5 pts.) Identify the critical numbers of g in the domain.

Critical numbers are x -values such that $g'(x) = 0$ or does not exist.

Given the graph, the critical numbers of g are approximately $x = -7$, $x = -5$, $x = -3$, and $x = -1$.

- (b) (5 pts.) State the absolute maximum and minimum **values** of $g(x)$ (if any).

The function g has an absolute maximum value of 4 which occurs when $x = -1$. There is no absolute minimum since g is undefined at $x = -7$.

- (c) (5 pts.) State the **x -values** at which $g(x)$ has a local maximum and minimum.

The function g has a local minimum at $x = -3$ and local maximums at $x = -1$ and $x = -5$.

5. (20 pts.) Find the linearization at $x = 1$ of $h(x) = 4\arctan(x^2) - \frac{\ln(x)}{x}$ and use it to approximate $h(1.01)$.

To find the linearization at $x = 1$, we need $h(1)$ and $h'(1)$.

$$\bullet h(1) = 4\arctan(1^2) - \frac{\ln(1)}{1} = 4\left(\frac{\pi}{4}\right) - 0 = \pi$$

$$\bullet h'(x) = \frac{4}{1+(x^2)^2} \cdot (2x) - \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \frac{8x}{1+x^4} - \frac{1-\ln(x)}{x^2}$$

$$h'(1) = \frac{8(1)}{1+1^4} - \frac{1-\ln(1)}{1^2} = 4 - 1 = 3$$

So, the linearization is $L(x) = 3(x - 1) + \pi$.

$$\Rightarrow h(1.01) \approx L(1.01) = 3(.01) + \pi = .03 + \pi$$