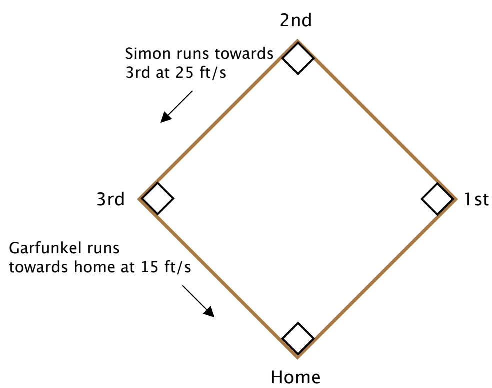
Math 124
Exam 3 Solutions

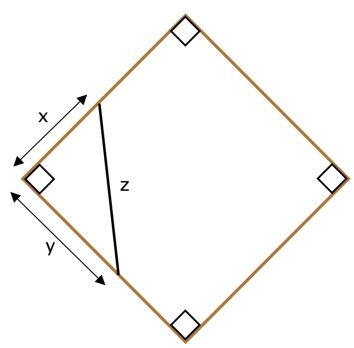
1. (25 pts.) Simon is on second base and Garfunkel is on third base during a baseball game. When the batter hits the ball, Simon and Garfunkel simultaneously start running towards the next bases. Simon runs at a speed of 25 feet/second and Garfunkel runs at a speed of 15 feet/second. How fast is the distance between them changing when Simon has run 50 feet and Garfunkel has run 30 feet? Is the distance between them increasing or decreasing? (Recall: The sides of a baseball diamond are 90 feet in length.)



x = distance between Simon and 3rd base
y = distance between Garfunkel and 3rd
z = distance between Simon and Garfunkel

$\Rightarrow \frac{dx}{dt} = -25$ ft/sec & $\frac{dy}{dt} = 15$ ft/sec
Unknown: $\frac{dz}{dt}$ when $x = 90 - 50 = 40$ and $y = 30$

Given our assignment of variables, we have the following diagram and equation to solve:



Equation: $x^2 + y^2 = z^2$

Differentiating with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$
$$\Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

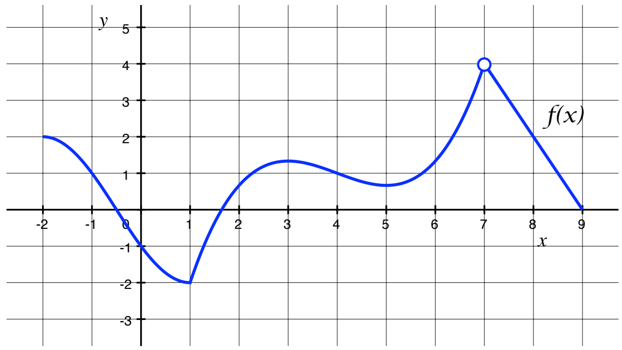
Note that if $x = 40$ and $y = 30$, then $z = \sqrt{40^2 + 30^2} = 50$ feet.

Plugging in values at the time at which $x = 40$ and $y = 30$:

$$40(-25) + 30(15) = 50\frac{dz}{dt}$$
$$\Rightarrow \quad \frac{dz}{dt} = -11 \text{ ft/sec}$$

So, the distance between Simon and Garfunkel is decreasing at a rate of 11 feet/sec.

2. (15 pts.) The graph of $f(x)$ on the interval $[-2, 9]$ is given below.



(a) (5 pts.) Approximate the critical numbers of f in the domain.

Critical numbers are x-values such that $f'(x) = 0$ or does not exist.

Given the graph, the critical numbers of f are approximately $x = 1$, $x = 3$, $x = 5$, and $x = 7$.

(b) (5 pts.) State the absolute maximum and minimum **values** of $f(x)$ (if any).

The function f has an absolute minimum value of -2 which occurs when $x = 1$. There is no absolute maximum since f is undefined at $x = 7$.

(c) (5 pts.) State the **x-values** at which $f(x)$ has a local maximum and minimum.

The function f has a local maximum at $x = 3$ and local minimums at $x = 1$ and $x = 5$.

3. (20 pts.) Find the linearization at $x = 0$ of $h(x) = x\ln(x^2+1) + 4\arccos(x)$ and use it to approximate $h(0.01)$.

To find the linearization at $x = 0$, we need $h(0)$ and $h'(0)$.

- $h(0) = 0 \cdot \ln(0^2+1) + 4\arccos(0) = 0 + 4\frac{\pi}{2} = 2\pi$
- $h'(x) = 1 \cdot \ln(x^2+1) + x \cdot \frac{2x}{x^2+1} - \frac{4}{\sqrt{1-x^2}} = \ln(x^2+1) + \frac{2x^2}{x^2+1} - \frac{4}{\sqrt{1-x^2}}$

  $h'(0) = \ln(0^2+1) + \frac{2 \cdot 0^2}{0^2+1} - \frac{4}{\sqrt{1-0^2}} = 0 + 0 - 4 = -4$

So, the linearization is $L(x) = -4x + 2\pi$.

$$\Rightarrow \quad h(.01) \approx L(.01) = -4(.01) + 2\pi = 2\pi - .04$$

4. (20 pts.) $g(x) = 6x^{1/3}(12 - x)$

(a) (12 pts.) Find the critical numbers g on the interval $[-1, 12]$.

Differentiating g using the product rule:

$$g'(x) = 6(\tfrac{1}{3})x^{-2/3}(12 - x) + 6x^{1/3}(-1)$$

$$= 2x^{-2/3}(12 - x) - 6x^{1/3}$$

$$= 24x^{-2/3} - 2x^{1/3} - 6x^{1/3}$$

$$= \frac{24}{x^{2/3}} \; -8x^{1/3}$$

Note that $g'(x)$ is undefined when $x = 0$, so $x = 0$ is a critical number.

Finding x such that $g'(x) = 0$:

$$\frac{24}{x^{2/3}} - 8x^{1/3} = 0$$

$$24 - 8x = 0 \quad \text{(Mult. both sides by } x^{2/3}\text{)}$$

$$\Rightarrow \quad x = 3$$

The function g has the critical numbers $x = 0$ and $x = 3$ on the interval $[-1, 12]$.

(b) (8 pts.) Find the absolute maximum and minimum **values** of g on the interval $[-1, 12]$.

Evaluating g at the endpoints and critical points:

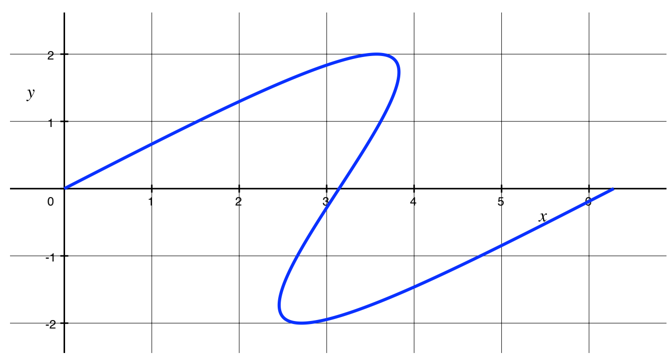
$g(-1) = -78$ $\qquad$ $g(0) = 0$

$g(12) = 0$ $\qquad$ $g(3) \approx 77.8814$

The absolute maximum value of g is approximately 77.8814 and it occurs at $x = 3$.
The absolute minimum value of g is -78 and it occurs at $x = -1$.

5. (20 pts.) The curve below is described by the equations $x(t) = t + 2\sin(t)$, $y(t) = 2\sin(t)$ for $0 \le t \le 2\pi$.

(a) (10 pts.) Find an equation of the tangent line to the curve at the point at which $t = \pi$.



To write an equation of the tangent, we need the point at which $t = \pi$ and the slope of the curve at $t = \pi$.

The point at which $t = \pi$ is $(x(\pi), y(\pi)) = (\pi + 2\sin(\pi), 2\sin(\pi)) = (\pi, 0)$.

The slope of the tangent line is given by $\frac{dy}{dx}$ at $t = \pi$.

Note that $x'(t) = 1 + 2\cos(t) \quad \Rightarrow \quad x'(\pi) = 1 + 2\cos(\pi) = 1 - 2 = -1$

and $y'(t) = 2\cos(t) \quad \Rightarrow \quad y'(\pi) = 2\cos(\pi) = -2$

$$\text{So, } \left.\frac{dy}{dx}\right|_{t=\pi} = \frac{y'(\pi)}{x'(\pi)} = \frac{-2}{-1} = 2.$$

Equation of tangent line: $\quad y = 2(x - \pi)$

(b) (10 pts.) Find the **values of t** at which the tangent line of the curve is vertical.

The tangent line will be vertical when $x'(t) = 0$.

Finding values of t at which $x'(t) = 0$:

$$1 + 2\cos(t) = 0 \quad \Rightarrow \quad \cos(t) = -\tfrac{1}{2}$$

$$\Rightarrow \quad t = \tfrac{2\pi}{3} \quad \text{or} \quad t = \tfrac{4\pi}{3}$$