Math 124
Exam 2 Solutions

1. (55 pts.) Evaluate the following.

   (a) (10 pts.) $\frac{d}{dx}[-3\ \sec^2(4-x)] = ?$

   $$\frac{d}{dx}[-3\ \sec^2(4-x)] = -3 \cdot 2\sec(4-x) \cdot \sec(4-x)\tan(4-x) \cdot (-1)$$
   (Chain rule $\times 2$)

   (b) (10 pts.) For $f(x) = \frac{1-7\ln(x)}{3x^2-x}$, find $f'(x)$.

   $$f'(x) = \frac{(3x^2-x)(-\frac{7}{x})-(1-7\ln(x))(6x-1)}{(3x^2-x)^2} \qquad \text{(Quotient Rule)}$$

   (c) (10 pts.) Find $\frac{dy}{dt}$ for $y = (\sin\ t)^{t+1}$.

   $$\ln\ y = \ln(\sin\ t)^{t+1} = (t+1)\ln(\sin\ t)$$

   Differentiating both sides with respect to t:
   $$\frac{1}{y} \cdot \frac{dy}{dt} = 1 \cdot \ln(\sin\ t) + (t+1)\left(\frac{1}{\sin\ t}\right) \cdot \cos\ t$$
   $$\Rightarrow \quad \frac{dy}{dt} = (\sin\ t)^{t+1}(\ln(\sin\ t) + (t+1) \cdot \cot\ t)$$

   (d) (8 pts.) Find the 43rd derivative of $\cos\ 2x$.

   Consider the first few derivatives of $\cos\ 2x$.

   1st: $\frac{d}{dx}(\cos\ 2x) = -2\sin\ 2x$
   2nd: $\frac{d}{dx}(-2\sin\ 2x) = -2^2\cos\ 2x$
   3rd: $\frac{d}{dx}(-2^2\cos\ 2x) = 2^3\sin\ 2x$
   4th: $\frac{d}{dx}(-2^3\sin\ 2x) = 2^4\cos\ 2x$

   The power of 2 keeps increasing, so by the 43rd derivative, the coefficient is 2^{43}. Also by the 43rd derivative, the derivative has cycled back to a positive $\sin\ 2x$. So, the 43rd derivative is $2^{43}\sin\ 2x$.

   (e) (7 pts.) $\frac{d}{dx}[\tan(\pi^2+1)] = ?$

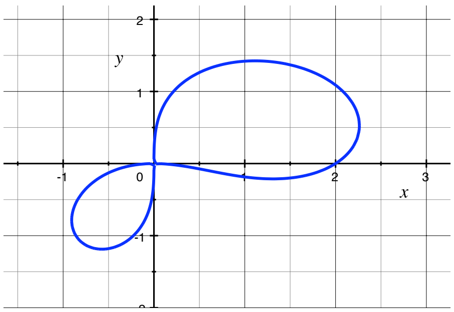
   Since $\tan(\pi^2+1)$ is a constant, $\frac{d}{dx}[\tan(\pi^2+1)] = 0$.

   (f) (10 pts.) For $g(x) = 2^{e^x}$, find $g''(x)$.

   $$g'(x) = 2^{e^x} \cdot \ln\ 2 \cdot e^x$$

$$g''(x) = \ln 2(2^{e^x}\ln 2 \cdot e^x \cdot e^x + 2^{e^x} \cdot e^x) \qquad \text{(Product Rule)}$$

2. (15 pts.) Find the slope of the curve $(x^2 + y^2)^2 = 5xy + 2x^3$ at the point $(2, 0)$.



$$\frac{d}{dx}((x^2+y^2)^2) = \frac{d}{dx}(5xy + 2x^3)$$
$$2(x^2+y^2)(2x + 2y\tfrac{dy}{dx}) = 5y + 5x\tfrac{dy}{dx} + 6x^2$$

Plugging in $x = 2$ and $y = 0$:
$$2(4+0)(4+0) = 0 + 10\tfrac{dy}{dx} + 24$$
$$8 = 10\tfrac{dy}{dx}$$

$$\Rightarrow \text{ Slope} = \frac{dy}{dx} = \frac{8}{10} = \frac{4}{5}$$

3. (15 pts.) $f(x) = \arcsin x = \sin^{-1}x$

(a) (5 pts.) For what x-value is $f(x) = \frac{\pi}{6}$?

If $f(x) = \frac{\pi}{6}$, then $\arcsin x = \frac{\pi}{6} \quad \Rightarrow \quad x = \sin\frac{\pi}{6} = \frac{1}{2}$.

(b) (10 pts.) Find an equation for the tangent line of f at the x-value found in part (a).

We first need to find the slope of f when $x = \frac{1}{2}$.
$$f'(x) = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad f'(\tfrac{1}{2}) = \frac{1}{\sqrt{1-(\frac{1}{2})^2}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$$

The tangent line has slope $\frac{2}{\sqrt{3}}$ and goes through the point $(\frac{1}{2}, \frac{\pi}{6})$.

Equation: $y - \frac{\pi}{6} = \frac{2}{\sqrt{3}}(x - \frac{1}{2})$

4. (15 pts.) The amount of water in a reservoir in millions of gallons is given by $w = f(t) = \sqrt{120t - 3t^2}$ on a given day t with $0 \leq t \leq 40$.

(a) (5 pts.) How much water is in the reservoir on day 30?

$$f(30) = \sqrt{120(30) - 3(30^2)} = 30$$
$\Rightarrow$ There was 30 million gallons in the reservoir on day 30.

(b) (10 pts.) Is water flowing into or out of the reservoir on day 30 and at what rate? (Include units on the rate of water flow.)

$$f'(t) = \frac{120-6t}{2\sqrt{120t-3t^2}}$$
$$\Rightarrow \quad f'(30) = \frac{120-6(30)}{2\sqrt{120(30)-3(30^2)}} = -1$$

Since the derivative is negative, the water is flowing out of the reservoir on day 30 at a rate of 1 million gallons per day.