Math 124
Exam 1 Solutions

1. (25 pts.) Evaluate the following limits. Justify your answers.

(a) (6 pts.) $\lim\limits_{x\to 3^+} \frac{2x-6}{x^2-6x+9}$

$$\lim_{x\to 3^+} \frac{2x-6}{x^2-6x+9} = \lim_{x\to 3^+} \frac{2(x-3)}{(x-3)^2} = \lim_{x\to 3^+} \frac{2}{x-3}$$

As x approaches 3 from the right, $x-3>0$ since $x>3$. Also, $x-3\to 0$ as $x\to 3$.

Thus, we must have that $\lim\limits_{x\to 3^+} \frac{2x-6}{x^2-6x+9} = \lim\limits_{x\to 3^+} \frac{2}{x-3} = \infty$.

(b) (6 pts.) $\lim\limits_{z\to 2^+} \frac{|4-z^2|}{4-z^2}$

Note that
$$|4-z^2| = \begin{cases} 4-z^2 & \text{if } 4-z^2\geq 0 \\ -(4-z^2) & \text{if } 4-z^2<0 \end{cases}$$

As z approaches 2 from the right, $4-z^2<0$ since $z>2$,
so $\lim\limits_{z\to 2^+} \frac{|4-z^2|}{4-z^2} = \lim\limits_{z\to 2^+} \frac{-(4-z^2)}{4-z^2} = \lim\limits_{z\to 2^+} -1 = -1$.

(c) (6 pts.) $\lim\limits_{t\to -\infty} \frac{2t^4+3t^3-1}{-4t^4-2t^2+t}$

- The degree of the polynomial in the numerator is the same as the degree of the polynomial in the denominator, so the limit as $t\to\infty$ will be the ratio of the coefficients.

$$\lim_{t\to -\infty} \frac{2t^4+3t^3-1}{-4t^4-2t^2+t} = -\frac{2}{4} = -\frac{1}{2}$$

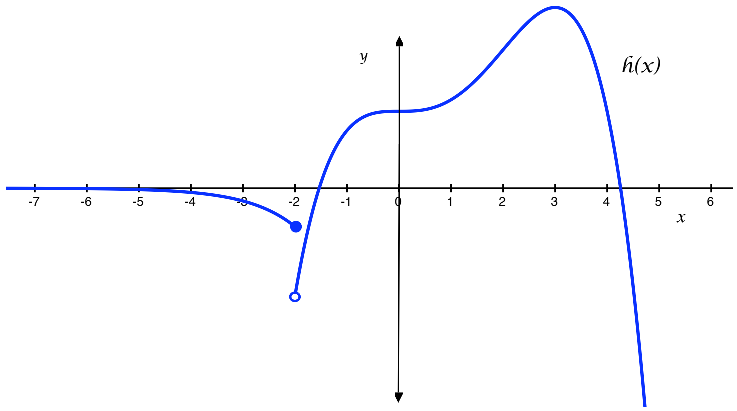
- If you would like to algebraically solve this limit, consider the following:

$$\begin{aligned}\lim_{t\to -\infty} \frac{2t^4+3t^3-1}{-4t^4-2t^2+t} &= \lim_{t\to -\infty} \frac{2+\frac{3}{t}-\frac{1}{t^4}}{-4-\frac{2}{t^2}+\frac{1}{t^3}} \quad \text{(Mult. num. and denom. by } \tfrac{1}{t^4}\text{)} \\ &= -\frac{2}{4} \\ &= -\frac{1}{2}\end{aligned}$$

(d) (7 pts.) $\lim\limits_{s\to\pi} \ln(\sin(s+\frac{\pi}{2}) + \frac{2\pi}{s})$

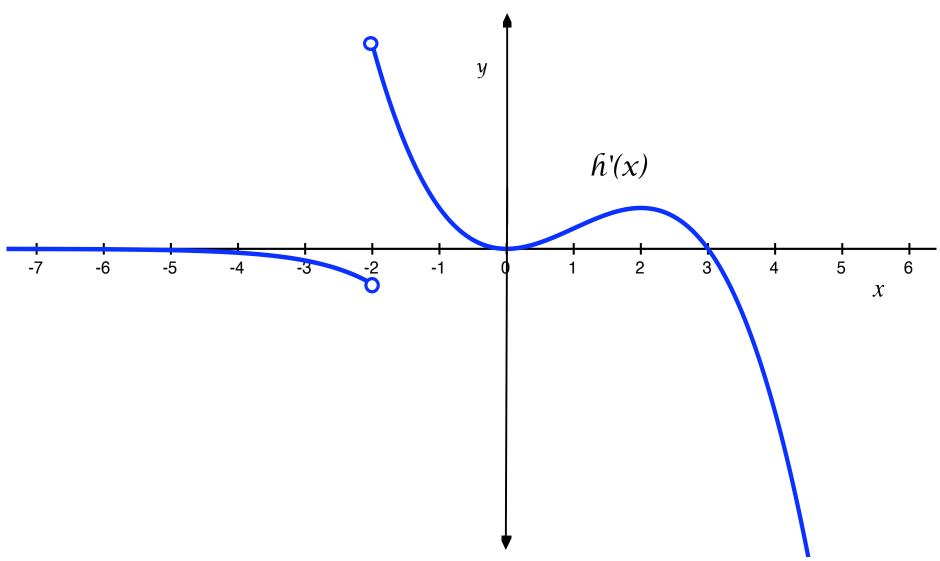
Since $\ln(\sin(s+\frac{\pi}{2}) + \frac{2\pi}{s})$ is continuous at $s=\pi$, we have that
$$\lim_{s\to\pi} \ln(\sin(s+\tfrac{\pi}{2}) + \tfrac{2\pi}{s}) = \ln(\sin(\pi+\tfrac{\pi}{2}) + \tfrac{2\pi}{\pi}) = \ln(-1+2) = \ln(1) = 0.$$

2. (15 pts.) For the following function h, sketch a graph of the derivative h' on the axis given below.



Note that $h'(x) = 0$ when $x = 0$ or $x = 3$.

- $\underline{x < -2}$: $h'(x) < 0$ and continues to become steeper and negative as x approaches -2. The function flattens out as x becomes more negative, so $h'(x)$ approaches 0 as x decreases.
- $\underline{-2 < x < 0}$: $h'(x) > 0$
- $\underline{0 < x < 3}$: $h'(x) > 0$
- $\underline{x > 3}$: $h'(x) < 0$ and continues to become steeper and negative as x increases.



3. (20 pts.)

$$g(x) = \begin{cases} \frac{4x^2+12x}{x^2+2x-3} & \text{if } x < -1 \\ 2x^2 & \text{if } -1 \leq x \leq 1 \\ e^{-x} - 1 & \text{if } x > 1 \end{cases}$$

(a) (6 pts.) Find the numbers at which g is discontinuous.

Note that the functions $2x^2$ and $e^{-x} - 1$ are continuous where they are defined. The function $\frac{4x^2+12x}{x^2+2x-3} = \frac{4x^2+12x}{(x+3)(x-1)}$ is undefined at $x = -3$ and $x = 1$ and thus discontinuous at those points. Since the function $g(x)$ is defined as $\frac{4x^2+12x}{x^2+2x-3}$ when $x < -1$, we have a discontinuity at $x = -3$.

The only other points for which there may be discontinuities are at $x = -1$ and $x = 1$.

- Note that $\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} \frac{4x^2+12x}{x^2+2x-3} = \frac{4(-1)^2+12(-1)}{(-1)^2+2(-1)-3} = 2$ and $\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} 2x^2 = 2(-1)^2 = 2.$

  Thus, $\lim_{x\to -1} g(x) = 2 = g(2)$. So, g is continuous at $x = -1$.

- Note that $\lim_{x\to 1^-} g(x) = \lim_{x\to -1^-} 2x^2 = 2(1)^2 = 2$ and $\lim_{x\to 1^+} g(x) = \lim_{x\to -1^+} e^{-x} - 1 = e^{-1} - 1 \neq 2.$

  Thus, $\lim_{x\to 1} g(x)$ does not exist. So, g is discontinuous at $x = 1$.

Discontinuities of g: $x = -3$ and $x = 1$

(b) (6 pts.) Evaluate $\lim_{x\to 0} g(x)$.

Since g is continuous at $x = 0$, $\lim_{x\to 0} g(x) = g(0) = 2(0)^2 = 0$.

(c) (8 pts.) Find the horizontal and vertical asymptotes (if any) for the function g.

- **Vertical Asymptotes**: The functions $2x^2$ and $e^{-x} - 1$ do not have vertical asymptotes for any x values. The function $\frac{4x^2+12x}{x^2+2x-3} = \frac{4x(x+3)}{(x+3)(x-1)} = \frac{4x}{x-1}$ (for $x \neq -3$) has a hole at $x = -3$ and a vertical asymptote at $x = 1$. However, the function g is defined as $\frac{4x^2+12x}{x^2+2x-3}$ only for $x < -1$, so g has no vertical asymptotes.

- **Horizontal Asymptotes**:
  * $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} e^{-x} - 1 = -1$ since $e^{-x} \to 0$ as $x \to \infty$.

  * $\lim_{x\to -\infty} g(x) = \lim_{x\to\infty} \frac{4x^2+12x}{x^2+2x-3} = 4$ (Same degree in num. and denom.).

  So, g has the horizontal asymptotes $y = -1$ and $y = 4$.

4. (20 pts.) Suppose the distance (in feet) of my pet turtle Sam from a certain point is given by the equation $s = 2 + \sqrt{t+1}$ at time t (in hours). What is Sam's instantaneous

velocity at 3 hours? (Use limits to evaluate and include units for the velocity.)

Two methods for calculation:

- Inst. vel. at $t = 3 = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$
$$= \lim_{h\to 0} \frac{2+\sqrt{3+h+1}-4}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$
$$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2}$$
$$= \frac{1}{4} \text{ feet/hour}$$

- Inst. vel. at $t = 3 = \lim_{t\to 3} \frac{f(t)-f(3)}{t-3}$
$$= \lim_{t\to 3} \frac{2+\sqrt{t+1}-4}{t-3}$$
$$= \lim_{t\to 3} \frac{\sqrt{t+1}-2}{t-3}$$
$$= \lim_{t\to 3} \frac{\sqrt{t+1}-2}{t-3} \cdot \frac{\sqrt{t+1}+2}{\sqrt{t+1}+2}$$
$$= \lim_{t\to 3} \frac{t+1-4}{(t-3)(\sqrt{t+1}+2)}$$
$$= \lim_{t\to 3} \frac{1}{\sqrt{t+1}+2}$$
$$= \frac{1}{4} \text{ feet/hour}$$

5. (20 pts.) Find the derivative of $f(x) = \frac{2}{x} + x^2$ using limits.

Two methods for calculation:

- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$$= \lim_{h\to 0} \frac{\frac{2}{x+h}+(x+h)^2-(\frac{2}{x}+x^2)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{x+h}-\frac{2}{x}}{h} + \frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2x-2(x+h)}{(x+h)x}}{h} + \frac{x^2+2xh+h^2-x^2}{h}$$
$$= \lim_{h\to 0} \frac{-2h}{h(x+h)x} + \frac{2xh+h^2}{h}$$
$$= \lim_{h\to 0} \frac{-2}{(x+h)x} + 2x + h$$
$$= -\frac{2}{x^2} + 2x$$

- $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$
$$= \lim_{x\to a} \frac{\frac{2}{x}+x^2-(\frac{2}{a}+a^2)}{x-a}$$
$$= \lim_{x\to a} \frac{\frac{2}{x}-\frac{2}{a}}{x-a} + \frac{x^2-a^2}{x-a}$$

$$
\begin{aligned}
&= \lim_{x\to a} \frac{\frac{2a-2x}{ax}}{x-a} + \frac{(x+a)(x-a)}{x-a} \\
&= \lim_{x\to a} \frac{-2(x-a)}{ax(x-a)} + x + a \\
&= \lim_{x\to a} -\frac{2}{ax} + x + a \\
&= -\frac{2}{a^2} + 2a
\end{aligned}
$$