Math 124
Exam 1 Solutions

1. (20 pts.)

$$f(x) = \begin{cases} e^x + 1 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 2 \\ \frac{3x^2-9x}{x^2-2x-3} & \text{if } x > 2 \end{cases}$$

(a) (6 pts.) Find the numbers at which f is discontinuous.

Note that the functions $\frac{1}{2}x^2$ and $e^x + 1$ are continuous where they are defined. The function $\frac{3x^2-9x}{x^2-2x-3} = \frac{3x^2-9x}{(x-3)(x+1)}$ is undefined at $x = 3$ and $x = -1$ and thus discontinuous at those points. Since the function $f(x)$ is defined as $\frac{3x^2-9x}{x^2-2x-3}$ when $x > 2$, we have a discontinuity at $x = 3$.

The only other points for which there may be discontinuities are at $x = 0$ and $x = 2$.

- Note that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x + 1 = e^0 + 1 = 2$
  and $\lim_{x\to 0^+} f(x) = \lim_{x\to -1^+} \frac{1}{2}x^2 = \frac{1}{2}(0)^2 = 0 \neq 2.$

  Thus, $\lim_{x\to 0} f(x)$ does not exist. So, f is discontinuous at $x = 0$.

- Note that $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{2}x^2 = \frac{1}{2}(2)^2 = 2$
  and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{3x^2-9x}{x^2-2x-3} = \frac{3(2)^2-9(2)}{(2)^2-2(2)-3} = \frac{-6}{-3} = 2$

  Thus, $\lim_{x\to 2} f(x) = 2 = f(2)$. So, f is continuous at $x = 2$.

Discontinuities of f: $x = 0$ and $x = 3$

(b) (6 pts.) Evaluate $\lim_{x\to 1} f(x)$.

Since f is continuous at $x = 1$, $\lim_{x\to 1} f(x) = f(1) = \frac{1}{2}(1)^2 = \frac{1}{2}$.

(c) (8 pts.) Find the horizontal and vertical asymptotes (if any) for the function f.

- **Vertical Asymptotes**: The functions $\frac{1}{2}x^2$ and $e^x + 1$ do not have vertical asymptotes for any x values. The function $\frac{3x^2-9x}{x^2-2x-3} = \frac{3x(x-3)}{(x-3)(x+1)} = \frac{3x}{x+1}$ (for $x \neq 3$) has a hole at $x = 3$ and a vertical asymptote at $x = -1$. However, the function f is defined as $\frac{3x^2-9x}{x^2-2x-3}$ only for $x > 2$, so f has no vertical asymptotes.

- **Horizontal Asymptotes**:
  * $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2-9x}{x^2-2x-3} = 3$ (Same degree in num. and denom.)

  * $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} e^x + 1 = 1$ since $e^x \to 0$ as $x \to -\infty$.

  So, f has the horizontal asymptotes $y = 1$ and $y = 3$.

2. (20 pts.) Suppose the distance (in feet) of a mechanical toy monkey from a certain point is given by the equation $s = \sqrt{t+4} + 3$ at time t (in seconds). What is the instantaneous velocity at 5 seconds? (Use limits to evaluate and include units for the velocity.)

   Two methods for calculation:

   - Inst. vel. at $t = 5 = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$
   $$\begin{aligned} &= \lim_{h\to 0} \frac{\sqrt{5+h+4}+3-6}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \\ &= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} \\ &= \frac{1}{6} \text{ feet/second} \end{aligned}$$

   - Inst. vel. at $t = 5 = \lim_{t\to 5} \frac{f(t)-f(5)}{t-5}$
   $$\begin{aligned} &= \lim_{t\to 5} \frac{\sqrt{t+4}+3-6}{t-5} \\ &= \lim_{t\to 5} \frac{\sqrt{t+4}-3}{t-5} \\ &= \lim_{t\to 5} \frac{\sqrt{t+4}-3}{t-5} \cdot \frac{\sqrt{t+4}+3}{\sqrt{t+4}+3} \\ &= \lim_{t\to 5} \frac{t+4-9}{(t-5)(\sqrt{t+4}+3)} \\ &= \lim_{t\to 5} \frac{1}{\sqrt{t+4}+3} \\ &= \frac{1}{6} \text{ feet/second} \end{aligned}$$

3. (20 pts.) Find the derivative of $g(x) = \frac{1}{2}x^2 + \frac{1}{x}$ using limits.

   Two methods for calculation:

   - $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$
   $$\begin{aligned} &= \lim_{h\to 0} \frac{\frac{1}{2}(x+h)^2 + \frac{1}{x+h} - (\frac{1}{2}x^2 + \frac{1}{x})}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x+h)^2 - \frac{1}{2}x^2}{h} + \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \end{aligned}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}x^2+xh+\frac{1}{2}h^2-\frac{1}{2}x^2}{h} + \frac{\frac{x-(x+h)}{(x+h)x}}{h}$$

$$= \lim_{h\to 0} \frac{xh+\frac{1}{2}h^2}{h} + \frac{-h}{h(x+h)x}$$

$$= \lim_{h\to 0} x+h + \frac{-1}{(x+h)x}$$

$$= x - \frac{1}{x^2}$$

- $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{2}x^2+\frac{1}{x}-(\frac{1}{2}a^2+\frac{1}{a})}{x-a}$

$$= \lim_{x\to a} \frac{\frac{1}{2}x^2-\frac{1}{2}a^2}{x-a} + \frac{\frac{1}{x}-\frac{1}{a}}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{2}(x+a)(x-a)}{x-a} + \frac{\frac{a-x}{ax}}{x-a}$$

$$= \lim_{x\to a} \frac{1}{2}(x+a)\ \frac{-(x-a)}{ax(x-a)}$$

$$= \lim_{x\to a} \frac{1}{2}(x+a)\ -\frac{1}{ax}$$

$$= a - \frac{1}{a^2}$$

4. (25 pts.) Evaluate the following limits. Justify your answers.

(a) (6 pts.) $\lim_{t\to 2^-} \frac{3t-6}{t^2-4t+4}$

$$\lim_{t\to 2^-} \frac{3t-6}{t^2-4t+4} = \lim_{t\to 2^-} \frac{3(t-2)}{(t-2)^2} = \lim_{t\to 2^-} \frac{3}{t-2}$$

As t approaches 2 from the left, $t-2<0$ since $t<2$. Also, $t-2\to 0$ as $t\to 2$.

Thus, we must have that $\lim_{t\to 2^-} \frac{3t-6}{t^2-4t+4} = \lim_{t\to 2^-} \frac{3}{t-2} = -\infty$.

(b) (6 pts.) $\lim_{x\to -\infty} \frac{-3x^5-x^3+1}{9x^5+2x^2-x}$

- The degree of the polynomial in the numerator is the same as the degree of the polynomial in the denominator, so the limit as $t\to\infty$ will be the ratio of the coefficients.

$$\lim_{x\to -\infty} \frac{-3x^5-x^3+1}{9x^5+2x^2-x} = -\frac{3}{9} = -\frac{1}{3}$$

- If you would like to algebraically solve this limit, consider the following:

$$\lim_{x\to -\infty} \frac{-3x^5-x^3+1}{9x^5+2x^2-x} = \lim_{x\to -\infty} \frac{-3-\frac{1}{x^2}+\frac{1}{x^5}}{9+\frac{2}{x^3}-\frac{1}{x^4}} \quad \text{(Mult. num. and denom. by } \tfrac{1}{x^5}\text{)}$$

$$= -\frac{3}{9}$$

$$= -\frac{1}{3}$$

(c) (6 pts.) $\lim\limits_{r\to 3^-} \frac{|r^2-9|}{r^2-9}$

Note that

$$|r^2 - 9| = \begin{cases} r^2 - 9 & \text{if } r^2 - 9 \geq 0 \\ -(r^2 - 9) & \text{if } r^2 - 9 < 0 \end{cases}$$

As r approaches 3 from the left, $r^2 - 9 < 0$ since $r < 3$,

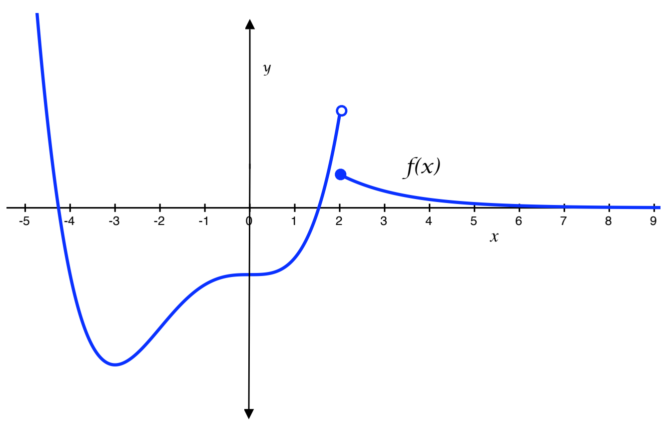
so $\lim\limits_{r\to 3^-} \frac{|r^2-9|}{r^2-9} = \lim\limits_{r\to 3^-} \frac{-(r^2-9)}{r^2-9} = \lim\limits_{r\to 3^-} -1 = -1.$

(d) (7 pts.) $\lim\limits_{s\to 1} \cos(\ln s - \frac{\pi}{s+2})$

Since $\cos(\ln s - \frac{\pi}{s+2})$ is continuous at $s = 1$, we have that

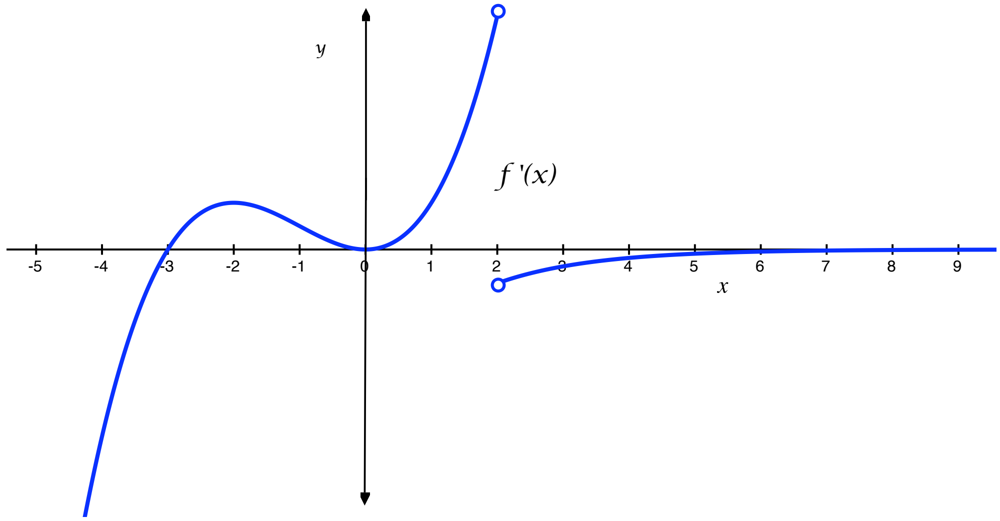
$\lim\limits_{s\to 1} \cos(\ln s - \frac{\pi}{s+2}) = \cos(\ln 1 - \frac{\pi}{1+2}) = \cos(-\frac{\pi}{3}) = \frac{1}{2}.$

5. (15 pts.) For the following function f, sketch a graph of the derivative f' on the axis given below.



Note that $f'(x) = 0$ when $x = 0$ or $x = -3$.

- $\underline{x < -3}$: $f'(x) < 0$ and becomes less steep and negative as x approaches -3.
- $\underline{-3 < x < 0}$: $f'(x) > 0$
- $\underline{0 < x < 2}$: $f'(x) > 0$
- $\underline{x > 2}$: $f'(x) < 0$. The function flattens out as x increases, so $f'(x)$ approaches 0 as x increases.


y
f '(x)
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
x