Math 151
Final Exam Answers

1. (a) $\lim_{x\to 2} \frac{\frac{1}{2}-\frac{2}{x^2}}{x-2} = \boxed{\frac{1}{2}}$

   (To evaluate, you can either use L'Hospital's Rule (indeterminate form "$\frac{0}{0}$") or simplify the expression by finding a common denominator for the difference in the numerator and factoring....)

   (b) $\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{\sqrt{9t^2-5}} = \boxed{\frac{1}{3}}$

   (Here's one way to evaluate: Multiply the numerator and denominator by $\frac{1}{t}$, this will give you

   $$\lim_{t\to\infty} \frac{\sqrt{1+\frac{1}{t^2}}}{\sqrt{9-\frac{5}{t^2}}} = \frac{\sqrt{1}}{\sqrt{9}} = \frac{1}{3}.)$$

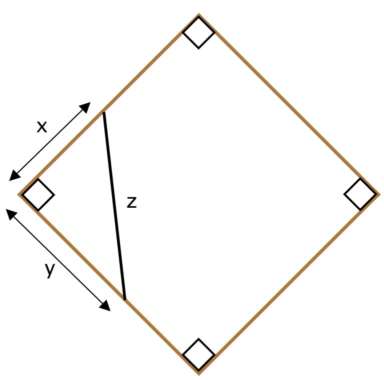
   (c) $\lim_{x\to 3^-} \frac{|x-3|}{x^2-6x+9} = \boxed{\infty}$

   (To evaluate, note that the limit simplifies to $\lim_{x\to 3^-} \frac{-1}{x-3}$ since $|x-3| = -(x-3)$ for $x < 3$. Note that the denominator is 0 at $x = 3$ while the numerator is not zero. This tells us that the limit must be infinite. Since the numerator is negative and for $x < 3$, the denominator is negative, we have that the limit must be positive infinity.)

   (d) $\lim_{\theta\to 0} \frac{\tan^2\theta}{\sin\theta} = \boxed{0}$

   (To evaluate, you can either use L'Hospital's Rule (indeterminate form "$\frac{0}{0}$") or simplify the expression by using the fact that $\tan x = \frac{\sin x}{\cos x}$.)

2. Here's one way to assign variables:



x = distance between Simon and 3rd base
y = distance between Garfunkel and 3rd
z = distance between Simon and Garfunkel

$\Rightarrow \frac{dx}{dt} = -25$ ft/sec & $\frac{dy}{dt} = 15$ ft/sec

Unknown: $\frac{dz}{dt}$ when $x = 90 - 50 = 40$ and $y = 30$

Equation: $x^2 + y^2 = z^2$

Differentiating with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \quad \Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

Note that if $x = 40$ and $y = 30$, then $z = \sqrt{40^2 + 30^2} = 50$ feet.

Plugging in values at the time at which $x = 40$ and $y = 30$:

$$40(-25) + 30(15) = 50\tfrac{dz}{dt} \qquad \Rightarrow \qquad \tfrac{dz}{dt} = -11 \text{ ft/sec}$$

$$\boxed{\text{The distance between Simon and Garfunkel is decreasing at a rate of 11 feet/sec.}}$$

3. (a) Left-side of equation: $2^3 + 4^3 = 72$
Right-side of equation: $9(2)(4) = 72$

So, the point $(2, 4)$ is on the curve since $x = 2$ and $y = 4$ satisfies the equation.

(b) Finding the slope at the point (2,4):

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[9xy]$$
$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 9y + 9x \cdot \frac{dy}{dx}$$

$$\Rightarrow \qquad 3(2)^2 + 3(4)^2 \frac{dy}{dx} = 9(4) + 9(2)\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{4}{5}$$

So the equation of the tangent line at $(2, 4)$ is $\boxed{y - 4 = \tfrac{4}{5}(x - 2).}$

(c) If P is a point near the point $(2, 4)$ on the curve, then x_1 can be approximated by using the tangent line from part (b) and finding the point on the line with y-coordinate equal to 4.1.

Plugging $y = 4.1$ into the equation of the tangent line we get

$$4.1 - 4 = \tfrac{4}{5}(x - 2) \qquad \Rightarrow \qquad x = 2.125$$

So, the point $(2.125, 4.1)$ in on the tangent line $\Rightarrow$ $\boxed{x_1 \approx 2.125}$

4. Using logarithmic differentiation and the product rule, we have that

$$\boxed{\frac{dy}{dt} = (\arcsin t)^{3t}[3\ln(\arcsin t) + 3t \cdot \frac{1}{\arcsin t} \cdot \frac{1}{\sqrt{1-t^2}}]}$$

5. (a) To be continuous at $t = 1$, we must have that $\lim_{t\to 1} h(t) = h(1)$. This means, we must have that the left and right-hand limits exist and are equivalent. So, $\lim_{t\to 1^-} h(t) = \lim_{t\to 1^+} h(t)$.

$$\lim_{t\to 1^-} \sqrt{24t - 20} + 3c = \lim_{t\to 1^+} 2t^2 + 2t + c$$

$$\sqrt{24 - 20} + 3c = 2 + 2 + c \quad \text{(Evaluating the limits)}$$
$$\Rightarrow \boxed{c = 1}$$

(b) In order for h to be differentiable at $t = 1$, we must have that the derivative exists at $t = 1$, that is, the derivative from the left and the right are equal at $t = 1$.

The derivative of $h(t)$ for $t < 1$ is $\dfrac{12}{\sqrt{24t-20}}$ so from the left, the derivative is $\dfrac{12}{\sqrt{24-20}} = 6$.

The derivative of $h(t)$ for $t > 1$ is $4t + 2$ so from the right, the derivative is $4+2 = 6$.

Since the left and right-hand derivatives match at $t = 1$, the function $\boxed{h \text{ is differentiable}}$ at $t = 1$.

6. (a) $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{1-2\sin(2t)}{\sin(2t)+2t\cos(2t)}$ Slope when $t = \pi$: $m = \frac{1-2\sin(2\pi)}{\sin(2\pi)+2\pi\cos(2\pi)} = \boxed{\frac{1}{2\pi}}$

(b) Horizontal Tangent $\Rightarrow$ $\frac{dy}{dx} = 0$ $\Rightarrow$ $y'(t) = 0$

So, we need to find values of t such that $1 - 2\sin(2t) = 0 \Rightarrow \sin(2t) = \frac{1}{2}$

$\Rightarrow 2t = \frac{\pi}{6}$ or $2t = \frac{5\pi}{6}$

So, $\boxed{t = \frac{\pi}{12} \text{ and } t = \frac{5\pi}{12}.}$

(Note: These are all the values between 0 and π since the period $y(t)$ is π.)

7. (a) First of all, note that the domain of $g(x)$ is $x > 0$.

Derivative: $g'(x) = \frac{x \cdot 2\ln x \cdot \frac{1}{x} - (\ln x)^2}{x^2} = \frac{2\ln x - (\ln x)^2}{x^2}$

Critical Points: Note that $g'(x)$ is defined for all values $x > 0$.

Also note that $g'(x) = 0 \Rightarrow 2\ln x - (\ln x)^2 = 0 \Rightarrow \ln x(2 - \ln x) = 0$

$\Rightarrow \ln x = 0$ or $\ln x = 2$

$\boxed{\text{So, the critical numbers are } x = 1 \text{ and } x = e^2.}$

- **First Derivative Test:**

Sign Chart:

g': $- - -$ 0 $+ + + +$ 0 $- - - -$

x: 0, 1, e^2

Evaluating $g'(x)$ at particular test x-values yields the above sign chart.

From the sign of the derivative we can see that g is increasing for the values $1 < x < e^2$. The function g is decreasing for $0 < x < 1$ and $e^2 < x$.

$\boxed{\text{So, } g(x) \text{ has a local max at } x = e^2 \text{ and a local min } x = 1.}$

- **Second Derivative Test:**

$g''(x) = \frac{x^2(\frac{2}{x} - \frac{2\ln x}{x}) - (2\ln x - (\ln x)^2)\cdot 2x}{x^4}$

Plugging in critical numbers: $g''(1) = 2 > 0 \Rightarrow$ Local min at $x = 1$

$g''(e^2) = -\frac{2}{e^6} < 0 \Rightarrow$ Local max at $x = e^2$

$\boxed{\text{So, } g(x) \text{ has a local max at } x = e^2 \text{ and a local min } x = 1.}$

(b) To find the absolute max and min values, evaluate $g(x)$ at the endpoints and the critical numbers.

$$g(1) = \frac{(\ln 1)^2}{1} + 12 = 12 \quad \leftarrow \quad \text{Absolute Minimum Value}$$

$$g(e^2) = \frac{(\ln e^2)^2}{e^2} + 12 = \frac{4}{e^2} + 12 \approx 12.5413 \quad \leftarrow \quad \text{Absolute Maximum Value}$$

$$g(20) = \frac{(\ln 20)^2}{20} + 12 \approx 12.4487$$

So, the $\boxed{\text{absolute maximum value is } \frac{4}{e^2} + 12}$ and the $\boxed{\text{absolute minimum value is 12.}}$

8. To find where $f(x)$ is concave up, we need to find values for which $f''(x) > 0$.

$$f'(x) = 5 + x^2 e^{\frac{1}{3}x^2} \quad \Rightarrow \quad f''(x) = 2xe^{\frac{1}{3}x^3} + x^2 \cdot x^2 e^{\frac{1}{3}x^3}$$

$$\Rightarrow \quad f''(x) = xe^{\frac{1}{3}x^3}(2 + x^3)$$

Note that we have possible inflection points where $f''(x)$ is 0 or undefined. $\Rightarrow \quad x = 0, x = \sqrt[3]{-2}$

Sign Chart:

f''	$+ + + +$	0	$- - - -$	0	$+ + + +$
x		$\sqrt[3]{-2}$		0	

Evaluating $f''(x)$ at particular test x-values yields the above sign chart.

So, the original function $f(x)$ is concave up for $\boxed{x < \sqrt[3]{-2} \text{ and } x > 0.}$

9. **Unknown:** Number $= x$ **Equation to Minimize:** Sum $= S(x) = x + \frac{1}{x}$

Finding the minimum: $S'(x) = 1 - \frac{1}{x^2}$ Critical Numbers: $x = 0,\ x = \pm 1$

Given that we are looking for values of x such that $x > 0$ (positive), we only have to consider the critical number $x = 1$.

To show that the sum is minimized at $x = 1$, consider the following sign chart for $S'(x)$:

Sign Chart:

S'		$- - - -$	0	$+ + + +$
x	0		1	

Evaluating $S'(x)$ at particular test x-values yields the above sign chart.

This shows that for positive numbers x, $S(x)$ decreases for $0 < x < 1$ and increases afterwards. Thus, the sum must have an absolute minimum when the number is $\boxed{x = 1.}$