Math 151
Final Exam Solutions

1. (a) Using the chain rule (Outer function: tangent, Inner function: $5x^2 \cdot \ln x$) and the product rule on the derivative of the inner function:

$$g'(x) = \sec^2(5x^2 \cdot \ln x) \cdot (10x \cdot \ln x + 5x^2 \cdot \frac{1}{x}) \quad + \quad 0 \qquad \text{(Note that 3ln(2) is a constant.)}$$

(b) We cannot use the power rule on this function since the exponent is not constant. We also cannot use the derivative of an exponential function since the base is not constant. This is a great problem for which to use logarithmic differentiation.

Taking the natural log of both sides of the equation:

$$\ln y = \ln((\arcsin t)^{3t})$$
$$\ln y = 3t \cdot \ln(\arcsin t)$$

Differentiating both sides with respect to t:

$$\frac{1}{y} \cdot \frac{dy}{dt} = 3 \cdot \ln(\arcsin t) \; + \; 3t \cdot \frac{1}{\arcsin t} \cdot \frac{1}{\sqrt{1-t^2}}$$

Multiplying both sides by $y = (\arcsin t)^{5t}$:

$$\frac{dy}{dt} = (\arcsin t)^{3t}[3 \cdot \ln(\arcsin t) \; + \; 3t \cdot \frac{1}{\arcsin t} \cdot \frac{1}{\sqrt{1-t^2}}]$$

(c) Let's look at the first few derivatives to see if there is a pattern.

Using the product rule: $f'(x) = e^x + xe^x$

$$f''(x) = e^x + e^x + xe^x \; = \; 2e^x + xe^x$$

$$f'''(x) = 2e^x + e^x + xe^x \; = \; 3e^x + xe^x$$

$$f^{(4)}(x) = 3e^x + e^x + xe^x \; = \; 4e^x + xe^x$$

Each time we take the derivative of xe^x, we get $e^x + xe^x$ (an additional term e^x). So, the 53th derivative is $f^{(53)}(x) = 53e^x + xe^x$.

2. **Point:** $x(\frac{\pi}{12}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ $\quad \Rightarrow \quad$ Point at $t = \frac{\pi}{2}$: $(\frac{\sqrt{2}}{2}, -\frac{1}{2})$

$y(\frac{\pi}{12}) = -\cos(\frac{\pi}{3}) = -\frac{1}{2}$

Slope: $x'(t) = \cos(3t) \cdot 3 \qquad y'(t) = \sin(4t) \cdot 4 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{4\sin(4t)}{3\cos(3t)}$

So, the slope when $t = \frac{\pi}{12}$ is $m = \frac{4\sin(4(\frac{\pi}{12}))}{3\cos(3(\frac{\pi}{12}))} = \frac{4\sin(\frac{\pi}{3})}{3\cos(\frac{\pi}{4})} = \frac{4\sqrt{3}}{3\sqrt{2}}$

Equation: $y + \frac{1}{2} = \frac{4\sqrt{3}}{3\sqrt{2}}(x - \frac{\sqrt{2}}{2})$

3. If the position is given by a function $f(t)$, then the derivative $f'(t)$ is the velocity. That is, $f'(t) = 3t^2 - t^3$.

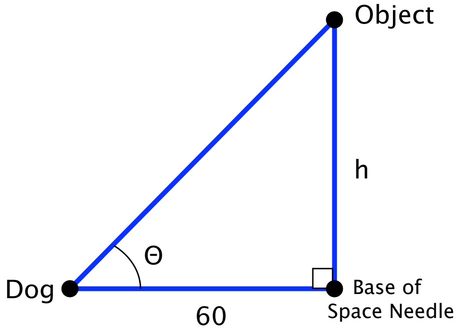
We know that $f(2) = 10$ and that $f'(2) = 2(2)^3 - 2^2 = 12$ ft/sec.

So, the linear approximation at $a = 2$ is
$$\begin{aligned} f'(x) &\approx f'(2)(x-2) + f(2). \\ &= 12(x-2) + 10 \end{aligned}$$

$$\text{Position at 2.1 seconds} = f(2.1) \approx 12(2.1 - 2) + 10 = 11.2 \text{ ft}$$

Another way to get to this approximation without the formal notation:
The velocity at 2 seconds is 12 feet per second. So, in 0.1 seconds, the raccoon would move approximately $12(0.1) = 1.2$ feet. Thus, at time 2.1 seconds, the approximate position of the raccoon is 11.2 feet.

4. Here is a diagram of the situation at some arbitrary time.



Quantities:
θ = angle of elevation
h = height of object

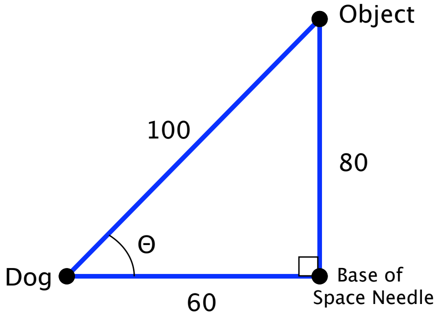
Rates:
$\frac{d\theta}{dt} = ?$ when $\frac{dh}{dt}$ = -40 m/s and h= 80 m

(Note: $\frac{dh}{dt}$ is negative since the height h is decreasing as the object falls.)

Equation: $\tan\theta = \frac{h}{60}$

Differentiating both sides with respect to t: $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{60} \cdot \frac{dh}{dt}$

Here is a diagram reflecting the moment at which $h = 80$:



Note that the distance between the dog and the object is 100 meters (using the Pythagorean Theorem).

So, when $h = 80$ m, $\sec\theta = \frac{100}{60} = \frac{5}{3}$.

(You can also find $\sec\theta$ by using $\tan\theta = \frac{80}{60}$ and finding θ or by using $1 + \tan^2\theta = \sec^2\theta$.)

Plugging in Values: $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{60} \cdot \frac{dh}{dt} \quad \Rightarrow \quad (\frac{5}{3})^2 \cdot \frac{d\theta}{dt} = \frac{1}{60} \cdot (-40) \quad \Rightarrow \quad \frac{d\theta}{dt} = -\frac{6}{25}$

So, the rate of change of the angle of elevation is $-\frac{6}{25}$ = -0.24 radians/sec.

5. To determine which graph is which function, note where each graph has slope 0. For example, graph a has slope 0 at the x-value for which graph c is equal to zero. So, graph c is a likely candidate for the derivative of graph a.

$\Rightarrow \quad f(x)$: Graph a $\qquad f'(x)$: Graph c $\qquad f''(x)$: Graph b

6. (a) This limit is of the indeterminate form "$\frac{0}{0}$".

Here are a couple of ways to do this limit:

- We can use L'Hospital's Rule on this limit: $\lim\limits_{t\to 2} \dfrac{t^5-32}{t^2-4} = \lim\limits_{t\to 2} \dfrac{5t^4}{2t} = \dfrac{5(2)^4}{2(2)} = 20$
- Factoring (not simple): $t^5 - 32 = (t-2)(t^4+2t^3+4t^2+8t+16)$ ← You can also get this using polynomial long division.

$$\Rightarrow \quad \lim_{t\to 2} \frac{t^5-32}{t^2-4} = \lim_{t\to 2} \frac{(t-2)(t^4+2t^3+4t^2+8t+16)}{(t-2)(t+2)}$$
$$= \lim_{t\to 2} \frac{t^4+2t^3+4t^2+8t+16}{t+2} = \frac{80}{4} = 20 \quad \text{(Plugging in } t=2\text{)}$$

(b) This function $\frac{\ln(x^2+2)}{e^{x+1}}$ is defined at $x=-1$, so $\lim\limits_{x\to\,-1} \dfrac{\ln(x^2+2)}{e^{x+1}} = \dfrac{\ln((-1)^2+2)}{e^{-1+1}}$

$$= \frac{\ln(1+2)}{e^0} = \ln(3)$$

(c) Note that as $x \to 3^-$, the numerator approaches 7 $\quad (2x+1 \to 7)$.

As $x \to 3^-$, the denominator approaches 0 $\quad (x(x-3)^2 \to 0)$. So, this limit is infinite.

Also note that since $x < 3$ (x is positive), we have that $x(x-3)^2$ is positive.

Since the numerator and denominator are both positive as $x \to 3^-$, $\lim\limits_{x\to 3^-} \dfrac{2x+1}{x(x-3)^2} = \infty$.

7. (a) **Derivative:** $\quad F'(x) = \frac{e^{2x}\cdot 9x^2 - 3x^3\cdot e^{2x}\cdot 2}{(e^{2x})^2} = \frac{e^{2x}(9x^2-6x^3)}{(e^{2x})^2} = \frac{9x^2-6x^3}{e^{2x}}$

Critical Points: Note that $F'(x)$ is defined for all values of x since e^{2x} is never zero.

Also note that $\quad F'(x) = \frac{9x^2-6x^3}{e^{2x}} = 0 \quad \Rightarrow \quad 9x^2 - 6x^3 = 0 \quad \Rightarrow \quad 3x^2(3-2x) = 0$

$$\Rightarrow \quad x = 0 \text{ and } x = \frac{3}{2}$$

So, the critical points are $x = 0$ and $x = \frac{3}{2}$.

- **First Derivative Test:**

Sign Chart:

F'	+ + + +	0	+ + + +	0	− − − −
x		0		$\frac{3}{2}$	

Plugging in x-values less than 0, between 0 and $\frac{3}{2}$, and greater than $\frac{3}{2}$ into F' will yield the above sign chart.

From the sign of the derivative we can see that F is increasing for the values $x < 0$ and $0 < x < \frac{3}{2}$. The function F is decreasing for $x > \frac{3}{2}$.

So, there is a local max at $x = \frac{3}{2}$ and $F(x)$ has neither a local max nor a local min at $x = 0$.

- **Second Derivative Test:**

$$F''(x) = \frac{e^{2x}(18x - 18x^2) - (9x^2 - 6x^3)e^{2x} \cdot 2}{(e^{2x})^2}$$

Plugging in critical numbers: $F''(\frac{3}{2}) \approx -0.08402 < 0 \quad \Rightarrow \quad$ Local max at $x = \frac{3}{2}$

$F''(0) = 0 \quad \Rightarrow \quad$ Inconclusive (Must use 1st deriv. test)

$\Rightarrow$ There is a local max at $x = \frac{3}{2}$ and $F(x)$ has neither a local max nor a local min at $x = 0$.

(b) To find the global max and min values, evaluate $F(x)$ at the endpoints and the critical numbers.

$$F(0) = \frac{3(0)^3}{e^{2(0)}} = 0 \quad \leftarrow \quad \text{Absolute Minimum Value}$$

$$F(3) = \frac{3(3)^3}{e^{2(3)}} = \frac{81}{e^6} \approx 0.20078$$

$$F(\tfrac{3}{2}) = \frac{3(\frac{3}{2})^3}{e^3} = \frac{10.125}{e^3} \approx 0.5041 \quad \leftarrow \quad \text{Absolute Maximum Value}$$

So, the absolute maximum value is $\frac{10.125}{e^3}$ and the absolute minimum value is 0.

8. (a) Note that the numerator and denominator are defined for all values, since they are exponential functions. The function $h(x)$ will be undefined only if the denominator is equal to 0.

$$e^x - 2 = 0 \quad \Rightarrow \quad e^x = 2 \quad \Rightarrow \quad x = \ln(2)$$

So, $h(x)$ is discontinuous at $x = \ln(2)$.

(b) **Vertical Asymptote(s):**
From part (a), we have that the denominator is equal to 0 at $x = \ln(2)$. Since the numerator is not equal to zero at $x = \ln(2)$, $h(x)$ has the vertical asymptote $x = \ln(2)$.

Horizontal Asymptote(s):
Consider the following limits: $\lim\limits_{x\to\infty} \frac{4e^x + 6}{e^x - 2}$ and $\lim\limits_{x\to -\infty} \frac{4e^x + 6}{e^x - 2}$

Here are a couple of ways to evaluate $\lim\limits_{x\to\infty} \frac{4e^x + 6}{e^x - 2}$. (Note: This is of the form "$\frac{\infty}{\infty}$".)

- Using L'Hospital's Rule: $\lim\limits_{x\to\infty} \frac{4e^x + 6}{e^x - 2} = \lim\limits_{x\to\infty} \frac{4e^x}{e^x} = \lim\limits_{x\to\infty} 4 = 4$
- $\lim\limits_{x\to\infty} \frac{4e^x + 6}{e^x - 2} = \lim\limits_{x\to\infty} \frac{4e^x + 6}{e^x - 2} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{4 + \frac{6}{e^x}}{1 - \frac{2}{e^x}} = \frac{4}{1} = 4$

To evaluate $\lim\limits_{x\to\,-\infty} \dfrac{4e^x+6}{e^x-2}$, note that $e^x \to 0$ as $x \to -\infty$.

So, $\lim\limits_{x\to\,-\infty} \dfrac{4e^x+6}{e^x-2} \;=\; \dfrac{4(0)+6}{0-2} \;=\; -3.$

Given these limits, $h(x)$ has the horizontal asymptotes $y = 4$ and $y = -3$.