Math 151
Exam 2 Solutions

1. (a) $\frac{d}{dt}[3\arctan(2^t)\cdot\sqrt[4]{t}] = \frac{d}{dt}[3\arctan(2^t)]\cdot\sqrt[4]{t} \; + \; 3\arctan(2^t)\cdot\frac{d}{dt}[\sqrt[4]{t}]$ (Product Rule)

$$= 3\cdot\frac{1}{1+(2^t)^2}\cdot 2^t\ln(2)\cdot\sqrt[4]{t} + 3\arctan(2^t)\cdot\tfrac{1}{4}x^{-3/4}$$ (Chain Rule for $3\arctan(2^t)$)

(b) $$g'(x) = \frac{\ln(x^3+4x)\frac{d}{dx}[8-3x^2] \; - \; (8-3x^2)\frac{d}{dx}[\ln(x^3+4x)]}{(\ln(x^3+4x))^2}$$ (Quotient Rule)

$$= \frac{\ln(x^3+4x)(-6x) \; - \; (8-3x^2)\cdot\frac{1}{x^3+4x}\cdot(3x^2+4)}{(\ln(x^3+4x))^2}$$

(c) First Derivative: $f'(x) = 2\cos(4x)\cdot(-\sin(4x))\cdot 4$ (Chain rule (twice))
$= -8\cos(4x)\sin(4x)$

Second Derivative: $$f''(x) = \frac{d}{dx}[-8\cos(4x)]\cdot\sin(4x) + (-8\cos(4x))\cdot\frac{d}{dx}[\sin(4x)]$$

$$= -8(-\sin(4x))\cdot 4\cdot\sin(4x) - 8\cos(4x)\cdot\cos(4x)\cdot 4$$

$$= 32\sin^2(4x) - 32\cos^2(4x)$$

(d) Since the derivative of e^x is e^x, the 80th derivative of $2e^x$ is $2e^x$ or $\frac{d^{80}}{dx^{80}}[2e^x] = 2e^x$.

Using the power rule, note that $\frac{d}{dx}[6x^{50}] = 6(50)x^{49}$, $\frac{d^2}{dx^2}[6x^{50}] = 6(50)(49)x^{48}$, ...

In fact, $\frac{d^{50}}{dx^{50}}[6x^{50}] = 6(50)(49)...(2)(1) = 6\cdot 50!$ $\leftarrow$ Constant!

So, $\frac{d^{51}}{dx^{51}}[6x^{50}] = 0$. All derivatives thereafter will also be 0.

Thus, $F^{80}(x) = 2e^x$.

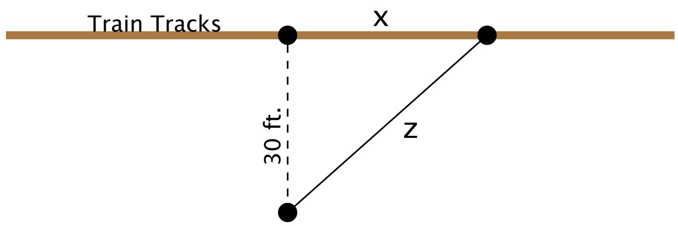
(e) This limit is of the indeterminate form "$\frac{0}{0}$", so it is a good idea to simplify the expression to evaluate.

$$\lim_{\theta\to 0}\frac{1-\sec\theta}{\cos\theta-1} = \lim_{\theta\to 0}\frac{1-\frac{1}{\cos\theta}}{\cos\theta-1} = \lim_{\theta\to 0}\frac{\frac{\cos\theta}{\cos\theta}-\frac{1}{\cos\theta}}{\cos\theta-1}$$

$$= \lim_{\theta\to 0}\frac{\frac{\cos\theta-1}{\cos\theta}}{\cos\theta-1}$$

$$= \lim_{\theta\to 0}\frac{1}{\cos\theta} = \frac{1}{\cos 0} = 1$$

2. The following is a figure at some time.



Quantities: x = distance train has gone past the point closest to you
z = distance between you and the train

Rates: $\frac{dx}{dt} = 35$ ft/sec $\qquad \frac{dz}{dt} = ?$ when $z = 50$

Equation: $x^2 + 30^2 = z^2 \quad \Rightarrow \quad x^2 + 900 = z^2$

Note that when $z = 50$, $x = 40$ $(x^2 + 900 = 50^2)$.

Differentiating both sides with respect to t:

$$\frac{d}{dt}[x^2 + 900] = \frac{d}{dt}[z^2]$$
$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

Plugging in known values: $2(40)(35) = 2(50)\frac{dz}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{2(40)(35)}{2(50)} = 28$ ft/sec

So, the distance between you and the train is increasing by 28 ft/sec at the moment at which the train is 50 feet away.

3. (a) Differentiate both sides with respect to x: $\frac{d}{dx}[e^x + 4x^2y] = \frac{d}{dx}[3\sin(y) + 1]$

$$e^x + 8x \cdot y + 4x^2 \cdot \frac{dy}{dx} = 3\cos(y) \cdot \frac{dy}{dx}$$

(Product rule on $4x^2y$)

$$e^x + 8xy = 3\cos(y)\frac{dy}{dx} - 4x^2\frac{dy}{dx}$$

$$e^x + 8xy = \frac{dy}{dx}(3\cos(y) - 4x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^x + 8xy}{3\cos(y) - 4x^2}$$

(b) The slope at (0,0) is $\frac{dy}{dx}\Big|_{x=0,y=0} = \frac{e^0 + 8(0)(0)}{3\cos(0) - 4(0)^2} = \frac{1}{3}$.

So, the equation of the tangent line is $y = \frac{1}{3}x$.

4. (a) **At rest:** We are looking for values of t for which $v = 0$. $\Rightarrow$

$$\frac{t^2}{4} - 3t = 0$$
$$t\left(\frac{t}{4} - 3\right) = 0$$

So, $t = 0$ or $\frac{t}{4} - 3 = 0 \Rightarrow t = 12$.

Thus, Sam is at rest at 0 and 12 minutes.

Moving in the negative direction:

We are looking for values of t for which $v < 0$. $\quad \Rightarrow \quad \dfrac{t^2}{4} - 3t < 0$

Here are a couple of ways to determine where $v < 0$:

- If $0 < t < 12$, then the factor t is positive and the factor $\dfrac{t}{4} - 3$ is negative, so $v < 0$.
  If $t > 12$, then both of the factors t and $\dfrac{t}{4} - 3$ are positive, so $v > 0$.

- Note that the graph of $v = \dfrac{t^2}{4} - 3t$ is a parabola that opens upward. Since it has zeros at $t = 0$ and $t = 12$, it is negative for $0 < t < 12$.

So, Sam is moving in the negative direction when $0 < t < 12$.

(b) Acceleration: $\quad a = \dfrac{dv}{dt} = \dfrac{2t}{4} - 3 = \dfrac{t}{2} - 3 \quad$ (The units are in/min^2.)

(c) Note that the velocity at $t = 4$ is negative (see part (a)). The acceleration at $t = 4$ is $a|_{t=4} = \dfrac{4}{2} - 3 = -1$.

Since both the acceleration and velocity are negative at 4 minutes, Sam is speeding up.