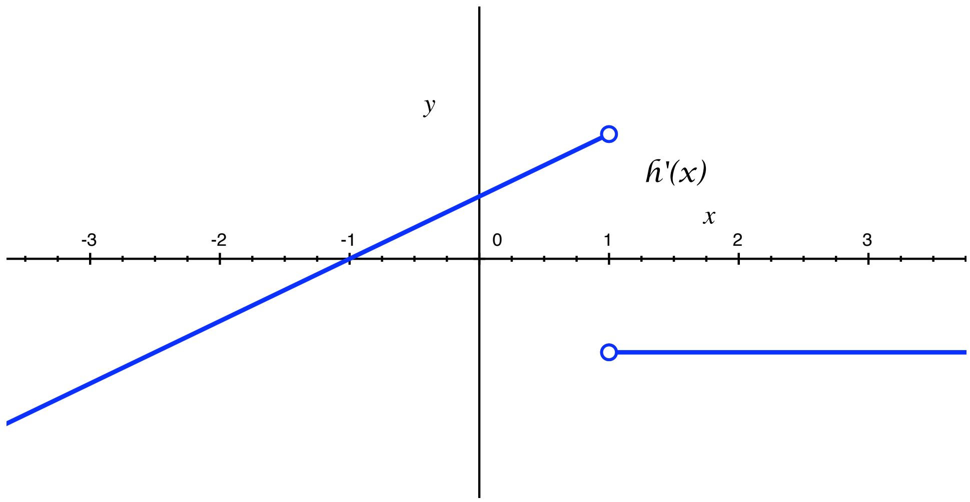
Math 151
Exam 1 Solutions

1. Note that the slope of $h(x)$ at $x = -1$ is 0 $\quad\Rightarrow\quad h'(-1) = 0$.

   Also, note that for $x < -1$, the slope is negative, and for $-1 < x < 1$, the slope is positive.

   For $x > 1$, the slope of $h(x)$ is constant and negative.

   Here is a possible sketch of $h'(x)$:



2. (a) The average velocity will be given by $\dfrac{\text{change in position}}{\text{change in time}}$.

   Since the position at $t = 0$ is given by $s = 6\sqrt{0} - 8(0) + 2 = 2$
   and the position at $t = 1$ is given by $s = 6\sqrt{1} - 8(1) + 2 = 0$,

   $$\text{the average velocity is } \frac{\text{change in position}}{\text{change in time}} = \frac{0-2}{1-0} = \text{-2 inches/minute}$$

   (b) The instantaneous velocity is given by
   $$\frac{ds}{dt} = \frac{d}{dt}[6t^{3/2} - 8t + 2] = 6(\tfrac{3}{2}t^{1/2}) - 8 = 9t^{1/2} - 8$$

   We are looking for t-value(s) at which $9t^{1/2} - 8 = 10 \quad\Rightarrow\quad 9t^{1/2} = 18$
   $$\Rightarrow\quad t^{1/2} = 2$$
   $$\Rightarrow\quad \sqrt{t} = 2$$
   $$\Rightarrow\quad t = 4 \quad \text{minutes}$$

3. (a)
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{9}{x+h+3} - 5(x+h) - (\frac{9}{x+3} - 5x)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{9}{x+h+3} - 5x - 5h - \frac{9}{x+3} + 5x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{9}{x+h+3} - \frac{9}{x+3}}{h} - \frac{5h}{h}$$
$$= \lim_{h\to 0} \frac{\frac{9(x+3)}{(x+h+3)(x+3)} - \frac{9(x+h+3)}{(x+h+3)(x+3)}}{h} - 5$$
$$= \lim_{h\to 0} \frac{\frac{9x+27-9x-9h-27}{(x+h+3)(x+3)}}{h} - 5$$
$$= \lim_{h\to 0} \frac{\frac{-9h}{(x+h+3)(x+3)}}{h} - 5$$
$$= \lim_{h\to 0} \frac{-9h}{(x+h+3)(x+3)} \cdot \frac{1}{h} - 5$$
$$= \lim_{h\to 0} \frac{-9}{(x+h+3)(x+3)} - 5$$
$$= \frac{-9}{(x+0+3)(x+3)} - 5$$
$$= \frac{-9}{(x+3)^2} - 5$$

(b) Slope at $x = 0$: $f'(0) = \frac{-9}{(0+3)^2} - 5 = -6$

Point when $x = 0$: $y = f(0) = \frac{9}{0+3} - 5(0) = 3 \qquad (0, 3)$

Equation of Line: $y = -6x + 3$

4. (a) Note that since this a piecewise function that is defined as $\frac{8x^2}{x^2+2x}$ for $x \leq 2$ and as $\frac{x^2}{x-1}$ for $x > 2$, there may be a discontinuity at $x = 2$.

There may also be discontinuities from each function on the domain given. These discontinuities will occur when the functions are undefined on the domain specified by the piecewise function.

- **Checking for continuity or discontinuity at $x = 2$:**

  Note that $g(2) = \frac{8(2)^2}{(2)^2+2(2)} = 4$.

  Also note that $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \frac{8x^2}{x^2+2x} = \frac{8(2)^2}{(2)^2+2(2)} = 4$

  and that $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{x^2}{x-1} = \frac{(2)^2}{2-1} = 4$.

  So, $\lim_{x\to 2} g(x) = 4 = g(2)$. By the definition of continuity, $g(x)$ is continuous at $x = 2$.

  (This may all look complicated, but we are just showing that the left and right functions meet at $x = 2$ (are equal at $x = 2$), so there is no jump or hole or vertical asymptote at $x = 2$.)

- **Checking for any discontinuities of $\frac{8x^2}{x^2+2x}$ for $x \leq 2$:**

  We can simplify the function: $\frac{8x^2}{x^2+2x} = \frac{8x^2}{x(x+2)} = \frac{8x}{x+2}$ for $x \neq 0$

Since this function is undefined at $x = 0$ and $x = -2$, $g(x)$ is discontinuous at these two values.
Given the simplified form, we can see that there is a vertical asymptote at $x = -2$, which means we have an infinite discontinuity at $x = -2$. We can also see that there is a hole at $x = 0$, since the factor of x in the denominator is divided out by a factor of x in the numerator. (See me for more details if needed.)

- **Checking for any discontinuities of $\frac{x^2}{x-1}$ for $x > 2$:**
  Since this function is only undefined at $x = 1$, which is not contained in the defined domain, $g(x)$ is continuous for $x > 2$.

So, $g(x)$ has a hole discontinuity at $x = 0$ and an infinite discontinuity at $x = -2$.

(b)
- $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^2}{x-1}$ (since for large x-values, $g(x) = \frac{x^2}{x-1}$)

  $= \infty$ since the degree in the numerator is greater than the degree in the denominator and both the numerator and denominator are positive

- $\lim_{x\to -\infty} g(x) = \lim_{x\to -\infty} \frac{8x^2}{x^2+2x}$ (since for large x-values, $g(x) = \frac{8x^2}{x^2+2x}$)

  $= 8$ since the degrees in the numerator and the denominator are equal

(c) From part (a), we have that $g(x)$ has the vertical asymptote $x = -2$. From part (b), we have that $g(x)$ has the horizontal asymptote $y = 8$.

5. (a) $\lim_{x\to 2^-} \frac{x^2-4}{3|x-2|} = \lim_{x\to 2^-} \frac{(x+2)(x-2)}{-3(x-2)}$ since $|x-2| = -(x-2)$ when $x < 2$

$$= \lim_{x\to 2^-} \frac{x+2}{-3}$$

$$= \frac{2+2}{-3} = -\frac{4}{3}$$

(b) $\lim_{t\to 0} \frac{e^{2t^2}}{t^2+5t+4} = \frac{e^{2(0)^2}}{(0)^2+5(0)+4} = \frac{1}{4}$

(c) $\lim_{x\to\infty} \frac{3}{\sqrt{x}-6} = 0$ since $\sqrt{x}-6 \to \infty$ as $x \to \infty$