

## Math 124 Worksheet #9 Solutions

1. If  $f(t) = (\tan t)^{2t}$ , what is the equation of the tangent line of  $f$  at the point  $(\frac{\pi}{4}, 1)$ ?

To get the derivative of  $f$ , I will use logarithmic differentiation because this is not an exponential function (the base is not constant) and I cannot use the power rule (the exponent is not constant).

Let  $y = (\tan t)^{2t}$ . Then  $\ln y = 2t \cdot \ln(\tan t)$ .

$$\begin{aligned}\Rightarrow \frac{d}{dx}[\ln y] &= \frac{d}{dx}[2t \cdot \ln(\tan t)] \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 2 \cdot \ln(\tan t) + 2t \cdot \frac{1}{\tan t} \cdot \sec^2 t \quad (\text{Product and Chain Rule}) \\ \Rightarrow f'(x) = \frac{dy}{dx} &= (\tan t)^{2t} [2 \cdot \ln(\tan t) + 2t \cdot \frac{1}{\tan t} \cdot \sec^2 t]\end{aligned}$$

Now for the rest of the problem:

$$\begin{aligned}\text{The slope of the tangent line at } (\frac{\pi}{4}, 1) \text{ is } f'(\frac{\pi}{4}) &= (\tan \frac{\pi}{4})^{2(\frac{\pi}{4})} [2 \cdot \ln(\tan \frac{\pi}{4}) + 2(\frac{\pi}{4}) \cdot \frac{1}{\tan \frac{\pi}{4}} \cdot \sec^2 \frac{\pi}{4}] \\ &= 1^{\frac{\pi}{2}} [2 \ln 1 + \frac{\pi}{2} \cdot 1 \cdot 2] \\ &= \pi\end{aligned}$$

So, the equation of the line is  $y - 1 = \pi(x - \frac{\pi}{4})$ .

2. Suppose the position of a particle is given by  $s = te^{3-t}$  in inches at  $t$  seconds.
- (a) When is the particle moving in the positive direction? Negative direction?

The velocity of the particle at time  $t$  is  $v(t) = e^{3-t} + te^{3-t}(-1) = e^{3-t}(1-t)$  in/sec.

The particle is moving in the positive direction when the velocity is positive.

Note that  $e^{3-t} > 0$  for all values of  $t$ .

$$\begin{array}{lll}\text{So, } v(t) > 0 \text{ when } 1 - t > 0 & \text{and} & v(t) < 0 \text{ when } 1 - t < 0 \\ \Rightarrow 1 > t & & \Rightarrow 1 < t\end{array}$$

Thus, the particle is moving in the positive direction when  $0 \leq t < 1$  seconds and it is moving in the negative direction when  $t > 1$  seconds.

- (b) When is the particle at rest?

The particle is at rest when  $v(t) = 0 \Rightarrow 1 - t = 0 \Rightarrow t = 1$  second.

- (c) Where is the particle located at 3 seconds?

The position is given by  $s = 3 \cdot e^{3-3} = 3$  inches.

- (d) What is the total distance traveled by the particle after 3 seconds? (Hint: See part (a).)

From time 0 to time 1, the particle is moving in the positive direction. At time 0, the position of the particle is 0 inches ( $s = 0 \cdot e^{3-0}$ ) and at time 1, the position of the particle is  $e^2 \approx 7.389$  inches ( $s = 1 \cdot e^{3-1}$ ). So, over the first second, the particle travels  $e^2$  inches.

From time 1 to time 3, the particle is moving in the negative direction. At time 1, the position of the particle is  $e^2 \approx 7.389$  inches. At time 3, the position of the particle is 3 inches. So, between time 1 and time 3, the particle traveled  $e^2 - 3 \approx 4.389$  inches in the negative direction.

So, from time 1 to time 3, the particle travels a total distance of  $e^2 + (e^2 - 3) = 2e^2 - 3 \approx 11.778$  inches.

3. Find the derivative of  $g(x) = \frac{\sin^3 x \cdot \sqrt{1+x}}{(2-5x)^3}$ .

Using logarithmic differentiation will make finding the derivative much easier.

$$\text{Let } y = \frac{\sin^3 x \cdot \sqrt{1+x}}{(2-5x)^3}. \quad \text{Then } \ln y = \ln \left[ \frac{\sin^3 x \cdot \sqrt{1+x}}{(2-5x)^3} \right].$$

$$\Rightarrow \ln y = 3 \ln(\sin x) + \frac{1}{2} \ln(1+x) - 3 \ln(2-5x)$$

$$\Rightarrow \frac{d}{dx} [\ln y] = \frac{d}{dx} [3 \ln(\sin x) + \frac{1}{2} \ln(1+x) - 3 \ln(2-5x)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 3 \cdot \frac{1}{\sin x} (\cos x) + \frac{1}{2} \cdot \frac{1}{1+x} - 3 \cdot \frac{1}{2-5x} (-5)$$

$$\frac{dy}{dx} = \frac{\sin^3 x \cdot \sqrt{1+x}}{(2-5x)^3} \left[ 3 \cot x + \frac{1}{2} \cdot \frac{1}{1+x} + \frac{15}{2-5x} \right]$$