Math 124 Worksheet #8 Solutions

1. Find the equation of the tangent line to the curve $x^2 - \sin(xy) + y^3 = 1$ at the point $(1, 0)$.

   Differentiating both sides with respect to x:

$$\frac{d}{dx}(x^2 - \sin(xy) + y^3) = \frac{d}{dx}(1)$$
$$2x - \cos(xy)(1 \cdot y + x\tfrac{dy}{dx}) + 3y^2\tfrac{dy}{dx} = 0 \quad \text{(Chain Rule \& Product Rule)}$$

   Plugging in $x = 1$ and $y = 0$:

$$2(1) - \cos(1 \cdot 0)(1 \cdot 0 + 1 \cdot \tfrac{dy}{dx}) + 3(0)^2\tfrac{dy}{dx} = 0$$
$$2 - 1 \cdot (0 + \tfrac{dy}{dx}) = 0$$
$$2 - \tfrac{dy}{dx} = 0$$

$$\text{So, } \frac{dy}{dx} = 2.$$

   Thus, an equation of the tangent line is $y = 2(x - 1)$.

2. Find all of the points of the graph $3x^2 + 4y^2 + 3xy = 26$ where the tangent line is horizontal. (Hint: After differentiating, plug in $\frac{dy}{dx} = 0$ to simplify your expression.)

   Differentiating both sides w.r.t. x: $\frac{d}{dx}(3x^2 + 4y^2 + 3xy) = \frac{d}{dx}(26)$

$$6x + 8y\tfrac{dy}{dx} + 3 \cdot y + 3x \cdot \tfrac{dy}{dx} = 0 \quad \text{(Product Rule for } 3xy)$$

   Instead of solving for $\frac{dy}{dx}$, we just want to find values of x and y for which $\frac{dy}{dx} = 0$, so we can plug $\frac{dy}{dx} = 0$ into the equation.

$$\Rightarrow \quad 6x + 8y(0) + 3y + 3x(0) = 0 \quad \Rightarrow \quad 6x + 3y = 0$$
$$\Rightarrow \quad y = -2x$$

   So, if a point (x, y) is on the curve, it must satisfy the equation $3x^2 + 4y^2 + 3xy = 26$. If the slope of the curve at (x, y) is 0, then we must have that $y = -2x$. The values of x and y must satisfy both equations. To find the x and y values that satisfy these equations, it is easiest to plug $y = -2x$ into the first equation.

$$\Rightarrow \quad 3x^2 + 4(-2x)^2 + 3x(-2x) = 26$$
$$3x^2 + 16x^2 - 6x^2 = 26$$
$$13x^2 = 26$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

   Since $y = -2x$, we have that when $x = \sqrt{2}$, $y = -2\sqrt{2}$ and when $x = -\sqrt{2}$, $y = 2\sqrt{2}$. So, the points at which the curve has horizontal tangents are $(\sqrt{2}, -2\sqrt{2})$ and $(-\sqrt{2}, 2\sqrt{2})$.

3. Find the 20th derivative of $\ln x$. (Note: An alternative notation for the 20th derivative of $\ln x$ is $D^{20} \ln x$.)

Consider the first few derivatives of $f(x) = \ln x$.
1st Derivative: $f'(x) = \frac{1}{x} = x^{-1}$
2nd Derivative: $f''(x) = (-1)x^{-2}$
3rd Derivative: $f'''(x) = (-1)(-2)x^{-3} = (-1)^2(1)(2)x^{-3}$
4th Derivative: $f^{(4)}(x) = (-1)(-2)(-3)x^{-4} = (-1)^3(1)(2)(3)x^{-4}$

In general: nth Derivative: $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$

The 20th derivative is $-19!x^{-20} = -\dfrac{19!}{x^{20}}$.

Notation: $19! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 18 \cdot 19$ $\qquad$ $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$

4. $\frac{d}{dx}[\ln(xe^x) + 8] = ?$

2 Options:

- $\frac{d}{dx}[\ln(xe^x) + 8] = \dfrac{1}{xe^x} \cdot (1 \cdot e^x + x \cdot e^x) + 0$ (Product Rule on derivative of xe^x)

  $= \frac{1}{x} + 1$

- Note that $\ln(xe^x) = \ln x + \ln e^x = \ln x + x$.

  So, $\frac{d}{dx}[\ln(xe^x) + 8] = \frac{d}{dx}(\ln x + x + 8) = \frac{1}{x} + 1$