Math 124 Worksheet #13 Solutions

1. Show that $f(x) = 2\cos x + 3x - 4$ has at exactly one real root. (Note: Proof by graphing in your calculator does not suffice.)

   Here are a couple of ways to show $f(x)$ has at least one real root:

   - Note that $f(-\pi) = 2\cos(-\pi) + 3(-\pi) - 4 = -3\pi - 6 < 0$
     and $f(\pi) = 2\cos(\pi) + 3(\pi) - 4 = 3\pi - 6 > 0$

     Since $f(x)$ is continuous and it takes on positive and negative values, the Intermediate Value Theorem tells us that for some value between pi and π, $f(x) = 0$.

   - Note that $\lim_{x\to\infty} 2\cos x + 3x - 4 = \infty$ because $3x \to \infty$ while $\cos x$ oscillates between -1 and 1.
     Another way to consider this limit is that $2\cos x + 3x - 4 \geq -2 + 3x - 4 = 3x - 6$.
     $\Rightarrow \quad \lim_{x\to\infty} 2\cos x + 3x - 4 \geq \lim_{x\to\infty} 3x - 6 = \infty$

     Similarly, $\lim_{x\to-\infty} 2\cos x + 3x - 4 = -\infty$. Again, since $f(x)$ is continuous, we have that $f(x)$ must have at least one root by the Intermediate Value Theorem.

   To show that $f(x)$ has **only** one real root:

   Assume that $f(x)$ has two roots, say $x = a$ and $x = b$. Then by Rolle's Theorem, there must be a point c such that $f'(c) = 0$.

   Note that $f'(x) = -2\sin x + 3$. If $f'(x) = 0$, then we must have that $\sin x = \frac{3}{2}$, which cannot be true. So, there are no values for which $f'(x) = 0$.
   $\Rightarrow \quad f(x)$ cannot have more than one real root.

2. Find a function $g(x)$ for which the derivative is $3x^2 + 12x + 9$.

   Since $\frac{d}{dx}[x^3] = 3x^2$, $\frac{d}{dx}[6x^2] = 12x$, and $\frac{d}{dx}[9x] = 9$, we have that

   $$g(x) = x^3 + 6x^2 + 9x + C, \text{ where } C \text{ is some constant.}$$

   (Answers for a particular function $g(x)$ will vary depending on the chosen value of C.)

3. On what intervals is the function $g(x)$ from problem 2 increasing?

   Note that $g'(x) = 3x^2 + 12x + 9 = 3(x + 3)(x + 1)$.

This is a quadratic with roots at $x = -3$ and $x = -1$. Since the graph is a parabola that opens upward, we have that $g'(x) > 0$ for $x > -1$ and $x < -3$ or on the intervals $(-\infty, -3)$ and $(-1, \infty)$.

So, $g(x)$ is increasing when $x < -3$ or when $x > -1$.