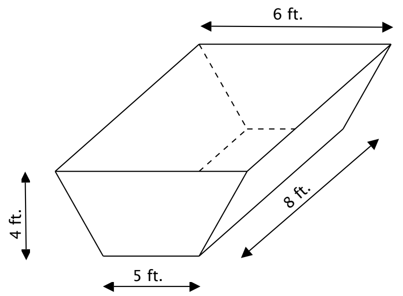
Calculus I - Math 124 Homework #7 Solutions

1. Water is being pumped at a rate of 4 ft^3/second into a trough that is 8 feet long with ends that have the shape of an isosceles trapezoid with dimensions as shown in the figure.



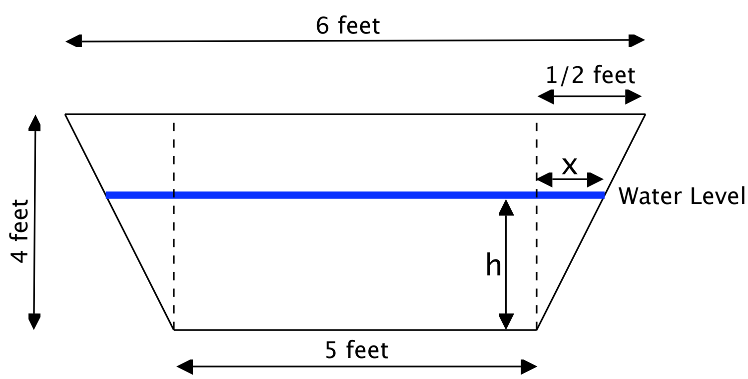
(a) How fast is the water level changing when the water level has a height of 2 feet?

Quantities: V = Volume of trough at time t
h = height of water at time t

Rates of Change: $\frac{dV}{dt} = 4 \text{ ft}^3/\text{sec}$ $\qquad \frac{dh}{dt} = ?$ when $h = 2$

The area of a trapezoid is $A = \frac{1}{2}h(B_1 + B_2)$ if it has height h and bases B_1 and B_2. Since the trough is 8 feet long, the volume of the trapezoidal trough is $V = \frac{1}{2}h(B_1 + B_2) \cdot 8 = 4h(B_1 + B_2)$.

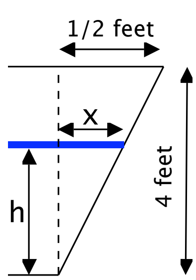
Consider the figure below and the quantities listed in the figure.



Given that $B_1 = 5$ and $B_2 = 5 + 2x$, we have that the volume of the water in the trough is

$$V = 4h(5 + 5 + 2x) = 4h(10 + 2x)$$

The problem with this formula is that we are interested in the quantities V and h and do not have the rate of change of x. However, using similar triangles, we can write x in terms of h.



We have $\frac{x}{h} = \frac{\frac{1}{2}}{4} \quad \Rightarrow \quad x = \frac{1}{8}h$

So, the volume of the trough is $V = 4h(10 + 2(\frac{1}{8}h)) = 40h + h^2$.

Taking the derivative of each side with respect to t:

$$\frac{d}{dt}[V] = \frac{d}{dt}[40h + h^2]$$
$$\frac{dV}{dt} = 40\frac{dh}{dt} + 2h\frac{dh}{dt}$$

Solving for $\frac{dh}{dt}$ when $h = 2$:

$$4 = 40\frac{dh}{dt} + 2(2)\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{11} \text{ ft/sec}$$

So, the water level is rising by $\frac{1}{11}$ ft/sec when the water level is 2 feet.

(b) How long will it take to fill up the trough?

The trough has volume $\frac{1}{2}(4)(5+6)\cdot 8 = 176$ ft^3. Since the water is being pumped in at a rate of 4 ft^3/sec, the trough will be filled after $\frac{176}{4} = 44$ seconds.

2. Find the linearization of $g(x) = 8\sqrt[4]{x+1}$ at $x = 15$ and use it to approximate $g(15.05)$.

The linearization of g at $x = 15$ is given by $L(x) = g'(15)(x - 15) + g(15)$.

- $g(15) = 8\sqrt[4]{15+1} = 8 \cdot 2 = 16$
- $g'(x) = 8 \cdot \frac{1}{4}(x+1)^{-3/4} = 2(x+1)^{-3/4}$
- $g'(15) = 2(15+1)^{-3/4} = \frac{1}{4}$

So, the linearization is $L(x) = \frac{1}{4}(x - 15) + 16$.

Approximating $g(15.05)$: $\quad g(15.05) \approx L(15.05) = \frac{1}{4}(15.05 - 15) + 16 = 16.0125$

3. Suppose we know that $h(2) = -3$ and $h'(x) = x^2 + 1$.

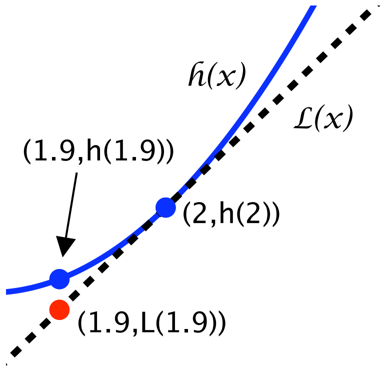
(a) Use linear approximation to estimate $h(1.9)$.

Linear Approximation: $\quad h(1.9) \approx h'(2)(1.9 - 2) + h(2) = 5(1.9 - 2) - 3 = -3.5$

(b) Is your estimate an over or underestimate? Explain.
(Hint: Consider the slope of $h(x)$ near $x = 2$.)

Note that $h'(x) < 5$ for $x < 2$ and near 2. $h'(x) > 5$ for $x > 2$ and near 2. So, the slope of $h(x)$ is positive and increasing near $x = 2$. (The tangent lines of $h(x)$ are getting steeper as x increases with x near 2.)

So, the graph of $h(x)$ near 2 and the tangent line at $x = 2$ look like the following (not to scale).



Given how the slopes of $h(x)$ are increasing, the tangent line will lie under the curve of $h(x)$.

So, the estimate from part (a) is an underestimate.

4. Find the nth derivative of $f(x) = \frac{x^n}{x-1}$ for some positive integer n.
(Hint: If your first few derivatives are looking scary, take a look at the original function and see if you can approach it differently.)

Note: If you use the quotient rule or produce rule on $f(x) = \frac{x^n}{x-1} = x^n(x-1)^{-1}$, you will find that your derivatives get messy rather quickly. Although, if you work to simplify your answers after each derivative using the quotient or product rule, you will find a nice pattern.

OR note that using long division, we have that $f(x) = x^{n-1} + x^{n-2} + x^{n-3} + ... + x + 1 + \frac{1}{x-1}$.

The nth derivative of the first n terms of f is 0, i.e., $\frac{d^n}{dx^n}[x^{n-1} + x^{n-2} + x^{n-3} + ... + x + 1] = 0$.

So, $f^{(n)}(x) = \frac{d^n}{dx^n}\left[\frac{1}{x-1}\right]$.

It is easier to find a pattern for the derivatives of $\frac{1}{x-1}$. Doing so, you should find that

$$f^{(n)}(x) = \frac{(-1)^n n!}{(x-1)^{n+1}}.$$