Math 124
Final Exam Solutions

1. (14 pts.) Evaluate the following limits. Justify your answers. If the limit is infinite, determine if it is $+\infty$ or $-\infty$.

(a) (5 pts.) $\lim_{x\to\infty}[2\arctan x - \frac{x^2-2x}{4x^2+5}] = \quad 2\cdot\frac{\pi}{2} - \frac{1}{4} \quad = \quad \pi - \frac{1}{4}$

since as $x \to \infty$, $\arctan x \to \frac{\pi}{2}$ and $\frac{x^2-2x}{4x^2+5} \to \frac{1}{4}$ (Degree in numerator = Degree in denominator)

(b) (5 pts.) $\lim_{t\to 1}\frac{t^3-1}{t^2-1}$

Here are a couple of ways (both involving polynomial division) to find this limit.

- Note that $\frac{t^3-1}{t^2-1} = t + \frac{t-1}{t^2-1}$ (Dividing t^3-1 by t^2-1)

$$= t + \frac{t-1}{(t-1)(t+1)}$$
$$= t + \frac{1}{t+1} \quad \text{for } t \neq 1$$

So, $\lim_{t\to 1}\frac{t^3-1}{t^2-1} = \lim_{t\to 1} t + \frac{1}{t+1} = 1 + \frac{1}{2} = \frac{3}{2}$.

- Note that $t=1$ is a zero for both the numerator t^3-1 and the denominator t^2-1. This means that $t-1$ must be a factor of both the numerator and the denominator. We can see that the denominator factors in the following way: $t^2-1=(t-1)(t+1)$

The numerator must factor in the following way: $t^3-1=(t-1)\cdot p(t)$ where $p(t)$ is a polynomial in t.

Using polynomial division, we get that $\frac{t^3-1}{t-1} = t^2+t+1$.

$$\Rightarrow \quad t^3-1=(t-1)\cdot(t^2+t+1)$$

So, $\lim_{t\to 1}\frac{t^3-1}{t^2-1} = \lim_{t\to 1}\frac{(t-1)(t^2+t+1)}{(t-1)(t+1)}$ (Factoring numerator and denominator)

$$= \lim_{t\to 1}\frac{t^2+t+1}{t+1}$$
$$= \frac{3}{2}$$

(c) (4 pts.) $\lim_{x\to\pi^+}\frac{\cos x}{(x-\pi)^3}$

Note that as $x \to \pi^+$, the numerator $\cos x$ approaches -1 and the denominator $(x-\pi)^3$ approaches 0.

Also note that $(x - \pi)^3 > 0$ for values of $x > \pi$.

Since the numerator is approaching -1 while the denominator is approaching zero positively,

$$\lim_{x \to \pi^+} \frac{\cos x}{(x - \pi)^3} = -\infty.$$

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2. (8 pts.) Find the intervals for which $h(x) = xe^{-\frac{1}{2}x^2}$ is increasing

First Derivative: $h'(x) = e^{-\frac{1}{2}x^2} + xe^{-\frac{1}{2}x^2}(-\frac{1}{2} \cdot 2x)$ (Product & Chain Rule)

$$= e^{-\frac{1}{2}x^2} - x^2 e^{-\frac{1}{2}x^2}$$
$$= e^{-\frac{1}{2}x^2}(1 - x^2)$$

Critical Numbers:
Note that $h'(x)$ is defined for all values of x.

Since $e^{-\frac{1}{2}x^2} > 0$ for all values of x, $h'(x) = 0$ when $1 - x^2 = 0$.
So, $h'(x) = 0$, when $x = \pm 1$.

Sign Chart:

h'	$- \ - \ - \ -$	0	$+ \ + \ + \ +$	0	$- \ - \ - \ -$
x		-1		1	

Plugging in x-values less than -1, between -1 and 1, and greater than 1 into h' will yield the above sign chart.

From the sign of the derivative we can see that h is increasing for the values $-1 < x < 1$.

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3. (7 pts.) Find the inflection point(s) of $y = \frac{3}{5}x^5 - x^4$.

First Derivative: $\frac{dy}{dx} = 3x^4 - 4x^3$

Second Derivative: $\frac{d^2y}{dx^2} = 12x^3 - 12x^2$

Note that $\frac{d^2y}{dx^2}$ is defined for all values of x and $\frac{d^2y}{dx^2} = 0$ at $x = 0$ and $x = 1$.
$(12x^3 - 12x^2 = 0 \quad \Rightarrow \quad 12x^2(x-1) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 1)$

Sign Chart:

$\frac{d^2y}{dx^2}$	$- \; - \; -$	0	$- \; - \; - \; -$	0	$+ \; + \; + \; +$
x		0		1	

Plugging in x-values less than 0, between 0 and 1, and greater than 1 into $\frac{d^2y}{dx^2}$ will yield the above sign chart.

From the sign of the second derivative we can see that $y = \frac{3}{5}x^5 - x^4$ only switches concavity at $x = 1$. Plugging in $x = 1$: $y = \frac{3}{5}(1)^5 - (1)^4 = -\frac{2}{5}$

So, the only inflection point is $(1, -\frac{2}{5})$.

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4. (15 pts.) Find the following. **You do not need to simplify your answers.**

(a) (5 pts.) Find $\frac{dy}{dx}$ for $y = [\ln(\tan x) + 2x]^2$

Using the chain and sum rules:

$$\frac{dy}{dx} = 2[\ln(\tan x) + 2x] \cdot (\frac{1}{\tan x} \cdot \sec^2 x + 2)$$

(b) (5 pts.) $\frac{d}{dx}[\sqrt[3]{2x^4 - 5} \cdot e^{\arcsin x}] = ?$

Using the product and chain rules:

$$\frac{d}{dx}[\sqrt[3]{2x^4 - 5} \cdot e^{\arcsin x}] = \frac{1}{3}(2x^4 - 5)^{-2/3}(8x^3) \cdot e^{\arcsin x} + \sqrt[3]{2x^4 - 5} \cdot e^{\arcsin x}(\frac{1}{\sqrt{1-x^2}})$$

(c) (5 pts.) Find $f'(t)$ when $f(t) = (\frac{2^t}{t} - t^2) \cdot \sec(3t)$.

Using the product, quotient, and chain rules:

$$f'(t) = (\frac{t \cdot 2^t \ln 2 - 2^t}{t^2} - 2t) \cdot \sec(3t) + (\frac{2^t}{t} - t^2) \cdot \sec(3t)\tan(3t) \cdot 3$$

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5. (12 pts.) Find the slope of the tangent line of the curve $e^{xy} + 4 = 5y - 3x$ at the point on the curve at which $x = 0$.

To find the point on the curve at which $x = 0$, we must find y when $x = 0$.

$$\Rightarrow \quad e^{0 \cdot y} + 4 = 5y - 3(0) \quad \Rightarrow \quad 1 + 4 = 5y \quad \Rightarrow \quad y = 1$$

So, we are looking for the slope of the tangent line at the point $(0, 1)$.

Implicit Differentiation: $$\frac{d}{dx}[e^{xy} + 4] = \frac{d}{dx}[5y - 3x]$$
$$e^{xy}(x \cdot \frac{dy}{dx} + y) = 5\frac{dy}{dx} - 3$$

Plugging in the values $x = 0$ and $y = 1$:

$$e^{0 \cdot 1}(0 \cdot \frac{dy}{dx} + 1) = 5\frac{dy}{dx} - 3$$
$$1 = 5\frac{dy}{dx} - 3 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{4}{5}$$

So, the slope is of the curve at the point $(0, 1)$ is $\frac{4}{5}$.

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6. (9 pts.) At exactly 5 seconds, the velocity of a particle is -10 cm/sec and the acceleration is -0.34 cm^2/sec.

   (a) (6 pts.) Approximate the velocity of the particle at 5.5 seconds using linear approximation.

   Note that $a(t)$, the acceleration at time t, is the rate of change of $v(t)$, the velocity at time t.

   Linear approximation: $$\begin{aligned} v(5.5) &\approx v(5) + v'(5)(5.5 - 5) \\ &= v(5) + a(5)(5.5 - 5) \\ &= -10 - 0.34(.5) \\ &= -10.17 \text{ cm/sec} \end{aligned}$$

   So, the velocity at 5.5 seconds is approximately -10.17 cm/sec.

   (b) (3 pts.) Is the particle speeding up or slowing down at 5 seconds?

   Since the velocity and acceleration have the same sign (they are both negative) at 5 seconds, the particle is speeding up, i.e., the absolute value of the velocity is increasing.

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7. (14 pts.) Find the minimum slope of the curve $y = \ln x + \frac{4}{3}x^3$. Show your work and justify that your solution is the absolute minimum.

Note that $y = \ln x + \frac{4}{3}x^3$ is defined only for $x > 0$.

The slope of the curve is given by $\frac{dy}{dx} = \frac{1}{x} + 4x^2$.

To find the absolute minimum slope, we must differentiate $\frac{dy}{dx}$ and find critical numbers of $\frac{dy}{dx}$.

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} + 8x$$

Note that this derivative is undefined at $x = 0$ (outside our domain) and it is equal to zero at $x = \frac{1}{2}$.

$$(-\frac{1}{x^2} + 8x = 0 \quad \Rightarrow \quad 8x = \frac{1}{x^2} \quad \Rightarrow \quad 8x^3 = 1 \quad \Rightarrow \quad x^3 = \frac{1}{8} \quad \Rightarrow \quad x = \frac{1}{2})$$

Sign Chart:

$\frac{d^2y}{dx^2}$	$- \; - \; - \; -$	0	$+ \; + \; + \; +$
x		$\frac{1}{2}$	

Plugging in x-values less than $\frac{1}{2}$ and greater than $\frac{1}{2}$ into $\frac{d^2y}{dx^2}$ will yield the above sign chart.

So, the slope $= \frac{dy}{dx}$ is decreasing for $0 < x < \frac{1}{2}$ and is increasing for $\frac{1}{2} < x$.

Thus, the slope must be at an absolute minimum when $x = \frac{1}{2}$.

So, the minimum slope is $\frac{dy}{dx}|_{x=1/2} = \frac{1}{\frac{1}{2}} + 4(\frac{1}{2})^2 = 2 + 1 = 3$.

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8. (12 pts.) For what values of x are the following functions continuous?

(a) (6 pts.) $g(x) = \begin{cases} \frac{-3x-3}{x^2-2x-3} & \text{if } x < 0 \\ \cos(\frac{x}{2}) & \text{if } x \geq 0 \end{cases}$

Note that $\frac{-3x-3}{x^2-2x-3} = \frac{-3(x+1)}{(x+1)(x-3)} = \frac{-3}{x-3}$ for $x \neq -1$. So for $x < 0$, $g(x)$ is discontinuous at $x = -1$.

Also note that $g(x)$ is continuous for $x > 0$ since $\cos(\frac{x}{2})$ is continuous for all values.

To check continuity at $x = 0$, we must evaluate the left and right-hand limits towards $x = 0$ and see if the limits match and are equal to $g(0)$. This is the same as evaluating the functions $\frac{-3x-3}{x^2-2x-3}$ and $\cos(\frac{x}{2})$ at $x = 0$.

Since $\cos(\frac{0}{2}) = 1$ and $\frac{-3(0)-3}{0^2-2(0)-3} = 1$, we have that the function $g(x)$ is continuous at $x = 0$.

So, $g(x)$ is continuous for all values except $x = -1$.

(b) (6 pts.) $f(x) = \ln(\sin^2 x) + \frac{1}{e^{x-2}}$

To find the values for which $f(x)$ is continuous, we must find where it is defined.

Since $\sin x$ is defined for all x-values and e^{x-2} is never zero, the only restrictions that can occur with the domain are values of x for which the $\ln(\sin^2 x)$ is undefined.

Natural log is defined only for strictly positive numbers. Since $\sin^2 x \geq 0$ for all x, the only values for which $\ln(\sin^2 x)$ is undefined are values for which $\sin^2 x = 0 \quad \Rightarrow \quad \sin x = 0.$

$\Rightarrow \quad x = k\pi$

(for an integer k)

So, $f(x)$ is continuous for all values except $x = k\pi$ for any integer k.

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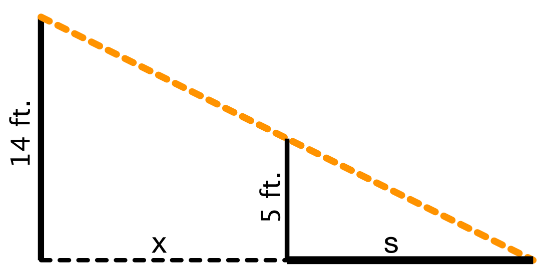
9. (10 pts.) A woman that is 5 feet tall is walking away from a 14 foot tall streetlight at a rate of 4 feet per second.

(a) (8 pts.) How fast is the **length of her shadow** changing when she is 10 feet away from the light? (Include units.)

Here is a diagram for the given situation.



Quantities:
Let x = the distance between the woman and the streetlight.

Let s = the length of the woman's shadow.

Rates of Change:

$\frac{dx}{dt} = 4$ ft/sec, $\qquad \frac{ds}{dt} = ?$ when $x = 10$

Equation relating x and s:

Using similar triangles, we have that $\quad \frac{x+s}{14} = \frac{s}{5}$

$$\Rightarrow \qquad 5x = 9s$$

Differentiating with respect to time:

$$\frac{d}{dt}[5x] = \frac{d}{dt}[9s]$$
$$5\frac{dx}{dt} = 9\frac{ds}{dt}$$

Plugging in known values:

$$5 \cdot 4 = 9\frac{ds}{dt} \qquad \Rightarrow \qquad \frac{ds}{dt} = \frac{20}{9}$$

Note: The rate of change $\frac{ds}{dt}$ is constant and does not depend on how far the woman is from the streetlight.

So, the length of the shadow is changing by $\frac{20}{9}$ feet/second.

(b) (3 pts.) Is the length of her shadow increasing or decreasing?

The shadow length is increasing. The rate of change is positive and does not depend on how far she is from the streetlight.