

**Math 124**  
**Exam 3 Solutions**

1. (10 pts.) Find a function  $g(x)$  for which  $g(1) = 3$  and  $g'(x) = \frac{2}{x} - 3$  for  $x > 0$ .

Since  $\frac{d}{dx}[2\ln x] = \frac{2}{x}$  and  $\frac{d}{dx}[-3x] = -3$ , we know that  $g(x) = 2\ln x - 3x + C$  where  $C$  is some constant.

To find  $C$ :  $3 = g(1) = 2\ln 1 - 3(1) + C = -3 + C \quad \Rightarrow \quad C = 6$

So,  $g(x) = 2\ln x - 3x + 6$ .

2. (16 pts.) Suppose that  $f(4) = 5$  and the **derivative** function is  $f'(x) = \frac{2}{\ln x}$ .

- (a) (10 pts.) Estimate  $f(4.1)$  using the tangent line approximation of  $f$  at  $x = 4$ .

Tangent line approximation of  $f$  at  $x = 4$ :

$$\begin{aligned} f(x) &\approx f(4) + f'(4)(x - 4) \quad \text{for } x \text{ near } 4 \\ &= 5 + \frac{2}{\ln 4}(x - 4) \end{aligned}$$

So,  $f(4.1) \approx 5 + \frac{2}{\ln 4}(4.1 - 4) \approx 5.1443$

- (b) (6 pts.) Is your estimate from part (a) an overestimate or underestimate? Explain.

The estimate is an overestimate.

Here are a couple of ways to think about why this is an overestimate.

- Note that the derivative  $f'(x) = \frac{2}{\ln x}$  is a decreasing function particularly around  $x = 4$  (As  $x$  gets larger,  $\ln x$  becomes larger, which will make  $\frac{2}{\ln x}$  decrease. You can also take a look at the graph of  $f'(x)$ .)

Since the derivative is decreasing, the graph of  $f(x)$  must be bending downwards (concave down) around  $x = 4$ . Since graph of  $f(x)$  is concave down, the tangent line at  $x = 4$  will be above the function for  $x$  near 4.

- We can take a look at the second derivative  $f''(x) = \frac{-2}{x(\ln x)^2}$ .

Note that  $f''(4) = \frac{-2}{4(\ln 4)^2} < 0$ , which tells us that  $f(x)$  is concave down at  $x = 4$ .

Since the graph of  $f(x)$  is concave down, the tangent line at  $x = 4$  will be above the function for  $x$  near 4.

3. (32 pts.)  $h(x) = 2\arctan x - \frac{x}{4}$  for all real values  $x$ .

(a) (15 pts.) Find the intervals on which  $h(x)$  is increasing and the intervals on which  $h(x)$  is decreasing. Give exact answers.

First Derivative:  $h'(x) = \frac{2}{1+x^2} - \frac{1}{4}$

**Critical Numbers:**

Since  $1 + x^2$  is never zero,  $h'(x)$  is defined for all  $x$ -values.

Finding  $x$ -values for which  $h'(x) = 0$ :  $\frac{2}{1+x^2} - \frac{1}{4} = 0$   
 $\frac{2}{1+x^2} = \frac{1}{4}$

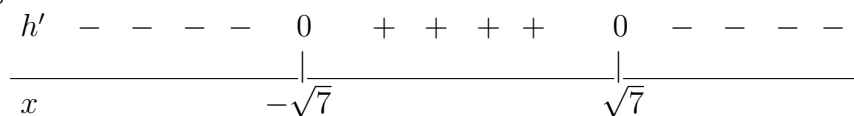
$$8 = 1 + x^2$$

$$7 = x^2$$

$$x = \pm\sqrt{7}$$

Critical Numbers:  $x = \sqrt{7}$ ,  $x = -\sqrt{7}$

Sign Chart:



Plugging in  $x$ -values less than  $-\sqrt{7}$ , between  $-\sqrt{7}$  and  $\sqrt{7}$ , and great than  $\sqrt{7}$  into  $h'$  will yield the above sign chart.

From the sign of the derivative we can see that  $h$  is increasing for the values  $-\sqrt{7} < x < \sqrt{7}$ . The function  $h$  is decreasing for  $x < -\sqrt{7}$  and  $x > \sqrt{7}$ .

(b) (9 pts.) What are the local maximum and minimum **values** of  $h(x)$ ?

Given the sign of the derivative  $h'$  in part (a), we can see that there is a local max at  $x = \sqrt{7}$  and a local min at  $x = -\sqrt{7}$ .

Local Max Value =  $h(\sqrt{7}) = 2\arctan\sqrt{7} - \frac{\sqrt{7}}{4} \approx 1.7574$

Local Min Value =  $h(-\sqrt{7}) = 2\arctan(-\sqrt{7}) + \frac{\sqrt{7}}{4} \approx -1.7574$

(c) (8 pts.) If the domain of  $h(x)$  is restricted to  $[-1, 5]$ , find the absolute maximum and minimum values of  $h$  on the interval.

Note that the only critical number in the domain is  $x = \sqrt{7}$ .

To find the absolute max and min values, compare the function values at the critical numbers and endpoints.

$$h(-1) = 2\arctan(-1) + \frac{1}{4} \approx -1.3208$$

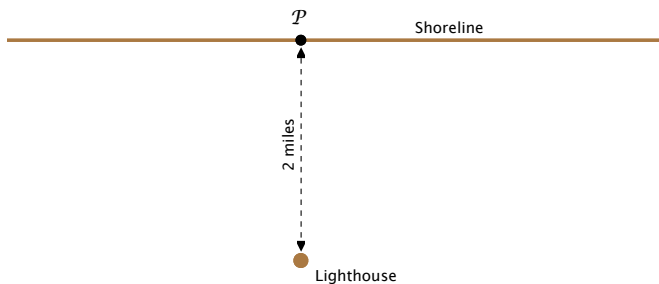
$$h(5) = 2\arctan 5 - \frac{5}{4} \approx 1.4968$$

$$h(\sqrt{7}) \approx 1.7574$$

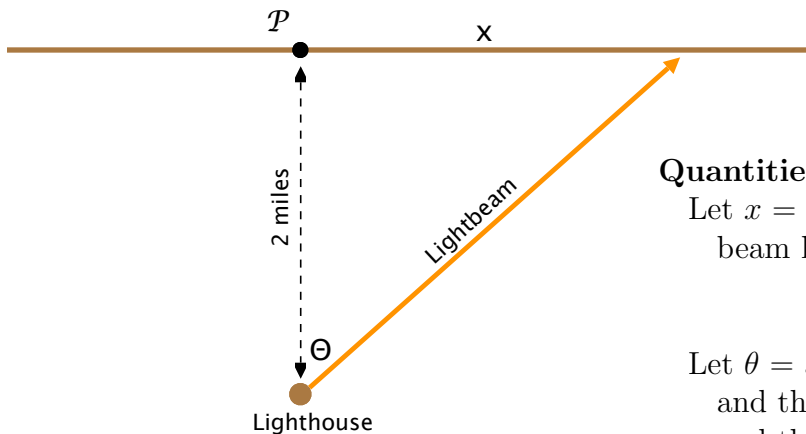
So, the absolute max value is approximately 1.7574, which occurs at  $x = \sqrt{7}$  and the absolute min value is approximately  $-1.3208$ , which occurs at  $x = -1$ .

4. (20 pts.) A lighthouse is located on a small island 2 miles away from the nearest point  $P$  on a straight shoreline. The light at the top of the lighthouse rotates at 3 revolutions per minute.

How fast is the light beam moving along the shoreline when it is 2 miles from  $P$ ? Include units in your answer.



Here is a similar diagram with a light beam hitting the shoreline.



**Quantities:**

Let  $x$  = distance from where the light beam hits the shore and the point  $P$

Let  $\theta$  = angle between the light beam and the line between the lighthouse and the point  $P$

**Rates of Change:**

$$\frac{d\theta}{dt} = 6\pi \text{ radians/minute (since the light rotates 3 rev/min)}$$

$$\frac{dx}{dt} = ? \text{ when } x = 2$$

**Equation Relating  $\theta$  and  $x$ :**

$$\tan \theta = \frac{x}{2} \quad \Rightarrow \quad 2 \tan \theta = x$$

**Differentiating with respect to  $t$ :**

$$\frac{d}{dt}[2 \tan \theta] = \frac{d}{dt}[x]$$

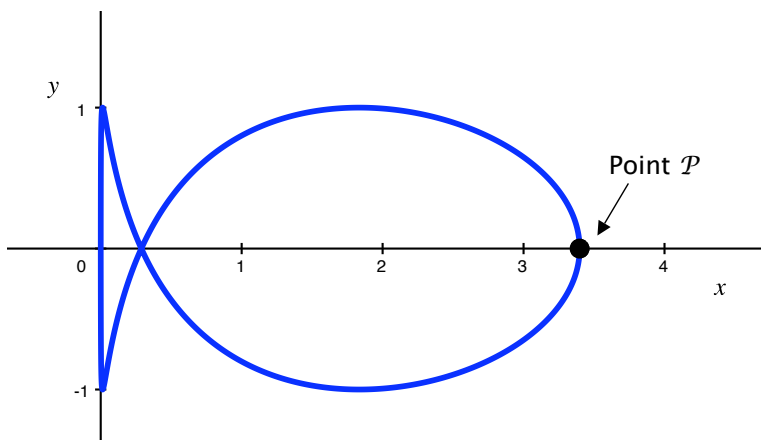
$$2 \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$$

**Plugging in known values to find  $\frac{dx}{dt}$  when  $x = 2$ :**

Note that when  $x = 2$ ,  $\tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

So, when  $x = 2$ ,  $2 \sec^2\left(\frac{\pi}{4}\right) \cdot 6\pi = \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 24\pi$  miles/min

5. (22 pts.) Consider the following parametric curve.



$$x(t) = \frac{5}{4}e^{1-t^2} \quad \text{for } -\pi \leq t \leq \pi.$$

$$y(t) = \sin(2t)$$

(a) (8 pts.) Find the  $x$ -coordinate of the point  $P$  as shown in the figure given that  $P$  is on the  $x$ -axis. Give the exact answer.

Here are a few ways to think about this problem.

- We can find  $t$ -values for which  $y(t) = 0$  by solving the equation  $\sin(2t) = 0$ .  

$$\sin(2t) = 0 \quad \Rightarrow \quad 2t = k\pi \quad \Rightarrow \quad t = k\left(\frac{\pi}{2}\right)$$

So in our domain for  $t$ ,  $y(t) = 0$  when  $t = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2},$  or  $\pi$ .

When plugging these values into  $x(t)$ , you will find that the largest value is  $x(0) = \frac{5}{4}e$  so this must be the  $x$ -coordinate of  $P$ .

- Note that at the point  $P$  we have a vertical tangent line  $\Rightarrow x'(t) = 0$ .

Since  $x'(t) = \frac{5}{4}e^{1-t^2}(-2t)$  is equal to 0 only at  $t = 0$ , we have that the vertical tangent must occur at  $t = 0$ .

So, the  $x$ -coordinate of the point is  $x(0) = \frac{5}{4}e$ .

- (b) (14 pts.) Find the slope of the curve when  $t = \frac{\pi}{2}$ . (You can leave your answer in exact form or round it to 4 decimal places.)

Derivatives:  $x'(t) = \frac{5}{4}e^{1-t^2}(-2t)$  and  $y'(t) = 2\cos(2t)$

$$\Rightarrow x'(\frac{\pi}{2}) = \frac{5}{4}e^{1-(\pi/2)^2}(-2(\frac{\pi}{2})) = -\frac{5}{4}\pi e^{1-(\pi/2)^2}$$

$$y'(\frac{\pi}{2}) = 2\cos(2(\frac{\pi}{2})) = -2$$

So, the slope at  $t = \frac{\pi}{2} = \frac{y'(\frac{\pi}{2})}{x'(\frac{\pi}{2})} = \frac{-2}{-\frac{5}{4}\pi e^{1-(\pi/2)^2}} = \frac{8}{5\pi e^{1-(\pi/2)^2}}$ .