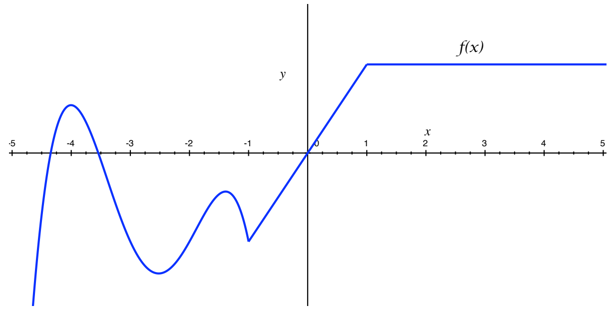
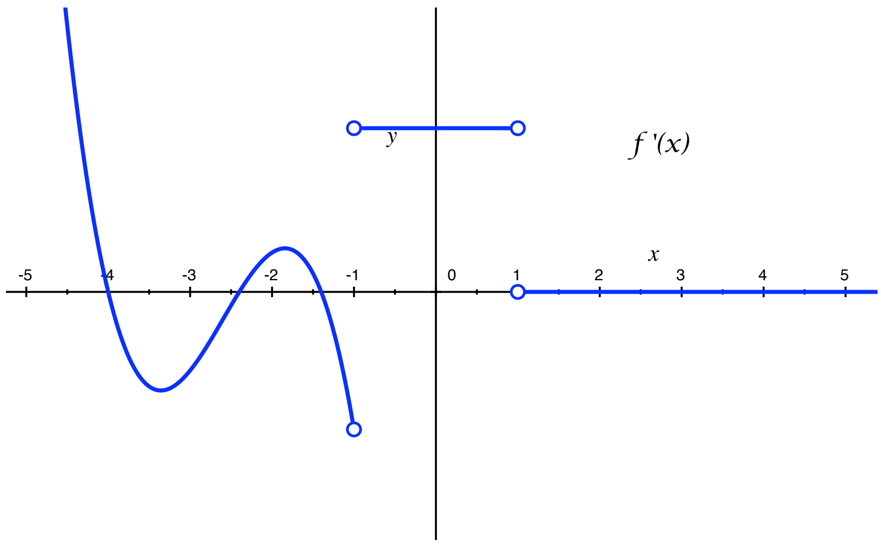
Math 124
Exam 1 Solutions

1. (15 pts.) For the following function f, sketch a graph of the derivative f' on the axis given below.





2. (27 pts.)

   (a) (6 pts.) Find the derivative of $h(x) = \dfrac{4}{x^2} - 2x + 7$.

   Note that $h(x) = \dfrac{4}{x^2} - 2x + 7 = 4x^{-2} - 2x + 7$.

   Using the power rule: $\quad h'(x) = 4(-2)x^{-3} - 2 + 0 = -\dfrac{8}{x^3} - 2$

   (b) (14 pts.) Verify the derivative of $h(x) = \dfrac{4}{x^2} - 2x + 7$ using the **limit definition** of the derivative.

   There are two limit definitions:

   - Provided the limit exists,
   $$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h} \qquad \text{(Sorry about all of the } h\text{'s)}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\frac{4}{(x+h)^2} - 2(x+h) + 7 - (\frac{4}{x^2} - 2x + 7)}{h} \\
&= \lim_{h\to 0} \frac{\frac{4}{(x+h)^2} - \frac{4}{x^2} - 2h}{h} \\
&= \lim_{h\to 0} \frac{\frac{4}{(x+h)^2} - \frac{4}{x^2}}{h} - \frac{2h}{h} \\
&= \lim_{h\to 0} \frac{\frac{4x^2-4(x+h)^2}{x^2(x+h)^2}}{h} - 2 \\
&= \lim_{h\to 0} \frac{-8xh - 4h^2}{hx^2(x+h)^2} - 2 \\
&= \lim_{h\to 0} \frac{-8x - 4h}{x^2(x+h)^2} - 2 \quad = \quad \frac{-8x}{x^4} - 2 = -\frac{8}{x^3} - 2
\end{aligned}
$$

- Provided the limit exists,

$$
\begin{aligned}
h'(a) &= \lim_{x\to a} \frac{h(x) - h(a)}{x - a} \\
&= \lim_{x\to a} \frac{\frac{4}{x^2} - 2x + 7 - (\frac{4}{a^2} - 2a + 7)}{x - a} \\
&= \lim_{x\to a} \frac{\frac{4}{x^2} - \frac{4}{a^2} - 2x + 2a}{x - a} \\
&= \lim_{x\to a} \frac{\frac{4}{x^2} - \frac{4}{a^2}}{x - a} - \frac{2x - 2a}{x - a} \\
&= \lim_{x\to a} \frac{\frac{4a^2-4x^2}{x^2a^2}}{x - a} - 2 \\
&= \lim_{x\to a} \frac{4(a-x)(a+x)}{x^2a^2(x-a)} - 2 \\
&= \lim_{x\to a} \frac{-4(a+x)}{x^2a^2} - 2 \quad = \quad \frac{-8a}{a^4} - 2 = -\frac{8}{a^3} - 2
\end{aligned}
$$

(c) (7 pts.) Find the equation of the tangent line of h at $x = 2$.

The slope of h at $x = 2$ is given by $h'(2) = -\frac{8}{2^3} - 2 = -3$.

The point of tangency is $(2, h(2)) = (2, 4)$.

Equation of the tangent line using point-slope form: $y - 4 = -3(x - 2)$
or $y = -3x + 10$

3. (24 pts.) Evaluate the following limits. Justify your answers. If the limit is infinite, determine if it is $+\infty$ or $-\infty$.

(a) (8 pts.) $\lim_{x\to 4} \frac{x^2-16}{x^3-3x^2-4x}$

$$\lim_{x\to 4} \frac{x^2-16}{x^3-3x^2-4x} = \lim_{x\to 4} \frac{(x+4)(x-4)}{x(x-4)(x+1)} = \lim_{x\to 4} \frac{x+4}{x(x+1)} = \frac{8}{20} = \frac{2}{5}$$

(b) (8 pts.) $\lim_{t\to\pi^+} \frac{t+5}{\sin\ t}$

Note that as $t \to \pi^+$, the numerator $t+5$ approaches $\pi + 5$ and the denominator $\sin\ t$ approaches 0.

Also note that $\sin\ t < 0$ for values of $t > \pi$ for t near π.

So, $\lim_{t\to\pi^+} \frac{t+5}{\sin\ t} = -\infty.$

(c) (8 pts.) $\lim_{x\to 3} [2x^2e^{x-3} + \ln(\frac{2}{x-1})]$

Since the function $[2x^2e^{x-3} + \ln(\frac{2}{x-1})]$ is continuous at $x = 3$ (it is defined at $x = 3$),

$$\begin{aligned}\lim_{x\to 3} [2x^2e^{x-3} + \ln(\tfrac{2}{x-1})] &= 2(3)^2e^{3-3} + \ln(\tfrac{2}{3-1}) \quad \text{(Evaluating function at } x = 3\text{)}\\ &= 18 + \ln 1\\ &= 18\end{aligned}$$

4. (18 pts.) $g(x) = \frac{2x+4}{x^2-x-6} - e^x + 4.$

(a) (7 pts.) Find the following limit: $\lim_{x\to\infty} g(x)$

Note that since the degree in the denominator of $\frac{2x+4}{x^2-x-6}$ is larger than in the numerator, as $x \to \infty$, $\frac{2x+4}{x^2-x-6} \to 0.$

Also note that as $x \to \infty$, $e^x \to \infty$ and $4 \to 4$.

So, $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{2x+4}{x^2-x-6} - e^x + 4 = -\infty.$

(b) (11 pts.) Find the **equations** of the horizontal and vertical asymptotes of $g(x)$.

To find the horizontal asymptotes, we must consider $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$. We already know that the first limit is infinite (part (a)), so no horizontal asymptote arises from $\lim_{x\to\infty} g(x)$.

$$\lim_{x\to-\infty} g(x) = \lim_{x\to\infty} \frac{2x+4}{x^2-x-6} - e^x + 4 = 4$$

since as $x \to -\infty$, $\frac{2x+4}{x^2-x-6} \to 0$, $e^x \to 0$, and $4 \to 4$.

So, $y = 4$ is a horizontal asymptote of $g(x)$.

To find any vertical asymptotes, we must first note that $-e^x + 4$ is defined for all real numbers, so if $g(x)$ has any vertical asymptotes, they must come from the vertical asymptotes of $\frac{2x+4}{x^2-x-6} = \frac{2(x+2)}{(x+2)(x-3)} = \frac{2}{x-3}$ for $x \neq -2$.

Since $\frac{2}{x-3}$ has a vertical asymptote at $x = 3$, $g(x)$ has the vertical asymptote $x = 3$.

5. (16 pts.) A particle is moving away from a certain point along a straight line and the distance (in cm) is given by $s = 3e^t - 8t$ at time t (in seconds).

   (a) (8 pts.) At what time (if any) is the velocity equal to 13 cm/sec?

   The velocity of the particle is given by $v = \frac{ds}{dt} = 3e^t - 8$.

   We are looking for t such that $v = 3e^t - 8 = 13 \quad \Rightarrow \quad e^t = 7$
   $\Rightarrow \quad t = \ln 7$ seconds

   (b) (8 pts.) What is the acceleration at 2 seconds? Include units.

   The acceleration of the particle is given by $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 3e^t$.

   At 2 seconds, the acceleration is $a = 3e^2 \ \frac{\text{cm}}{\text{sec}^2}$.