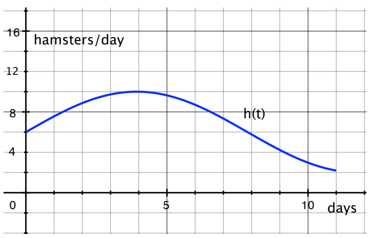
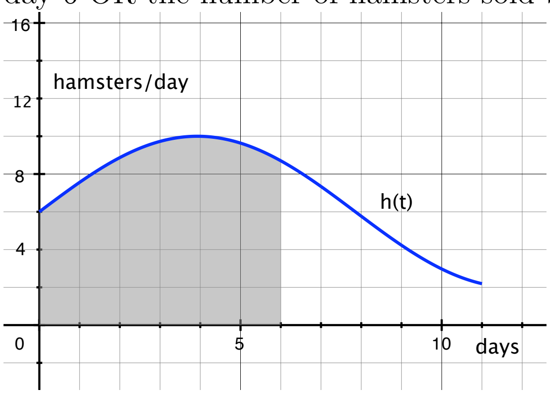
Math 112
Exam 2 Solutions

1. (15 pts.) At a local pet shop, the rate of change of sales of hamsters (in hamsters/day) on day t is given by the function $h(t)$. Approximate the number of hamsters sold between day 0 and day 6.



Since the function $h(t)$ = rate of change of sales of hamsters, the area under the curve from $t = 0$ to $t = 6$ is the total accumulated change of sales of hamsters from day 0 to day 6 OR the number of hamsters sold between day 0 and day 6.



Using the grid, there are approximately 26.5 boxes in the shaded area. Each box represents 2 hamsters sold $\Rightarrow$ approximately 53 hamsters were sold between day 0 and day 6.

2. (25 pts.)

   (a) (8 pts.) $\frac{d}{ds}\left(\frac{5s}{1+3s^2}\right)$

   Using the quotient rule,
$$\frac{d}{ds}\left(\frac{5s}{1+3s^2}\right) = \frac{(1+3s^2)\frac{d}{ds}(5s)-(5s)\frac{d}{ds}(1+3s^2)}{(1+3s^2)^2}$$
$$= \frac{(1+3s^2)(5)-(5s)(6s)}{(1+3s^2)^2}$$
$$= \frac{5+15s^2-30s^2}{(1+3s^2)^2}$$
$$= \frac{5-15s^2}{(1+3s^2)^2}$$

(b) (8 pts.) $\int 3x^5 + 9e^{3x} + 1\, dx$

$$\int 3x^5 + 9e^{3x} + 1\, dx = \frac{3}{6}x^6 + \frac{9}{3}e^{3x} + x + C$$
$$= \tfrac{1}{2}x^6 + 3e^{3x} + x + C$$

(c) (9 pts.) $\int \frac{6t^2}{(t^3+2)^3}\ dt$

Using integration by substitution, let $u = t^3 + 2 \quad \Rightarrow \quad du = 3t^2\ dt$

$$\int \frac{6t^2}{(t^3+2)^3}\ dt = \int \frac{2}{u^3}\ du = \int 2u^{-3}\, du$$
$$= \frac{2}{-2}u^{-2} + C = -\frac{1}{u^2}\ +C = -\frac{1}{(t^3+2)^2}\ +C$$

3. (15 pts.) Suppose the population of the United States in millions of people is modeled by the logistic function $P = \frac{548}{1+25e^{-.06t}}$ where t is in years since 1950.

(a) (5 pts.) What does this model predict for the maximum sustainable population in the US?

This model predicts that the maximum sustainable population in the US is 548 million people (numerator of function).

(b) (10 pts.) According to this model, during which year is the population growing the fastest?

The population is growing the fastest at the inflection point (point of diminishing returns) of the function. Since we have a logistic model, this point occurs when the population is half the maximum sustainable population, i.e., the population grows the fastest when $P = \frac{1}{2}(548) = 274$.

$$274 = \frac{548}{1+25e^{-.06t}} \quad \Rightarrow \quad 25e^{-.06t} = 1$$
$$\Rightarrow \quad e^{-.06t} = .04$$
$$\Rightarrow \quad -.06t = \ln(.04)$$
$$\Rightarrow \quad t \approx 53.65$$

So, according to this model, the population is growing the fastest in the 2003.

4. (25 pts.) $g(t) = (t-3)e^{-t} + 5$

(a) (15 pts.) Find the critical points of $g(t)$. Identify each critical point as a local maximum, local minimum, or neither using either the first or second derivative test.

First derivative of g: $g'(t) = 1 \cdot e^{-t} + (t-3) \cdot (-e^{-t})$
$= e^{-t}(1 - (t-3))$
$= e^{-t}(4 - t)$

Since e^{-t} is always positive and never zero, we have that the only critical point is at $t = 4$.

- Using the First Derivative Test:
  Again, $e^{-t} > 0$ for all t. For $t > 4$, $(4 - t)$ is negative and for $t < 4$, $(4 - t)$ is positive.
  So, for $t > 4$, $g'(t)$ is negative and for $t < 4$, $g'(t)$ is positive.
  This implies that there is a local maximum at $t = 4$.

- Using the Second Derivative Test:
  $g''(t) = -e^{-t} \cdot (4 - t) + e^{-t} \cdot (-1)$
  $= -e^{-t}(4 - t + 1)$
  $= -e^{-t}(5 - t)$

  Plugging in $t = 4$, $g''(4) = -e^{-4}(5 - 4) = -e^{-4} < 0$
  This implies that there is a local maximum at $t = 4$.

(b) (10 pts.) If the domain is restricted to $0 \le t \le 6$, what are the global maximum and global minimum values of g on this domain?

Evaluating g at the endpoints and critical points:
$g(0) = (0 - 3)e^0 + 5 = 2$ $\quad$ $g(4) = (4 - 3)e^{-4} + 5 = e^{-4} + 5 \approx 5.018$
$g(6) = (6 - 3)e^{-6} + 5 = 3e^{-6} + 5 \approx 5.007$

We have a global maximum at $t = 4$ with a value of $e^{-4} + 5 \approx 5.018$.
We have a global minimum at $t = 0$ with a value of 2.

5. (20 pts.) Suppose you sell trinkets for \$24 each. The following tables give the marginal cost and average cost for producing certain quantities of trinkets.

quantity	300	400	500	600	700	800	900	1000	1100	1200
MC	30	24	20	19	18	21	27	35	40	45

(a) (5 pts.) To increase profit, should you increase or decrease production at the production level of $q = 600$?

Note that we have $MR(q) = 24$ for all q.

Since $MR(600) > MC(600)$ you should increase production (Marginal profit is positive).

(b) (5 pts.) Estimate the production level that maximizes profit.

Profit is maximized at quantities for which $MR = MC$ and we are switching from $MR > MC$ to $MC > MR$.

Looking at the table, we can estimate that $MR = MC$ at $q = 400$ and $q = 850$. Since $MR < MC$ for $q < 400$, $MR > MC$ for $400 < q < 850$, and $MR < MC$ for $q > 850$, we have that at profit should be maximized if you produce around 850 trinkets.

quantity	300	400	500	600	700	800	900	1000	1100	1200
AC	26	25	24	23	21	20	22	23	25	27

(c) (5 pts.) If you sell 600 trinkets, is your profit positive or negative, i.e. are you making or losing money?

Since $MR(600) = 24$ and $AC(600) = 23$, your profit is positive ($MR > AC$).

(d) (5 pts.) Estimate the maximum profit.

From part (b), we estimated that profit is maximized if you produce 850 trinkets.

The total revenue from producing 850 trinkets is $850 \cdot 24 = \$20400$.
Estimating average cost, we have that $AC(850) \approx \$21$.
So, the approximate total cost of producing 850 trinkets is $850 \cdot 21 = \$17850$.

So, the maximum profit is approximately \$20400-\$17850=\$2550.