The Inflection Point of the Logistic Equation

The inflection point of the logistic equation $P(t) = \frac{L}{1+Ce^{-kt}}$ (for positive constants L, C, and k) occurs at a time t for which $P(t) = \frac{L}{2}$.

To prove this, consider the derivatives of $P(t)$ (At the inflection point, we must have that $P''(t) = 0$):

$$P'(t) = \frac{(1+Ce^{-kt})(0) - L(-kCe^{-kt})}{(1+Ce^{-kt})^2} = \frac{kCLe^{-kt}}{(1+Ce^{-kt})^2} \text{ (Quotient Rule)}$$

$$P''(t) = \frac{(1+Ce^{-kt})^2(-k^2CLe^{-kt}) \ - \ kCLe^{-kt}(2)(1+Ce^{-kt})(-kCe^{-kt})}{(1+Ce^{-kt})^4}$$

$$= \frac{k^2CLe^{-kt}(1+Ce^{-kt})[-(1+Ce^{-kt})+2Ce^{-kt}]}{(1+Ce^{-kt})^4}$$

$$= \frac{k^2CLe^{-kt}(1+Ce^{-kt})[-1+Ce^{-kt}]}{(1+Ce^{-kt})^4}$$

Since k^2CLe^{-kt} and $1+Ce^{-kt}$ are positive, nonzero quantities, $P''(t) = 0$ only if $-1+Ce^{-kt} = 0$.

Note that $-1 + Ce^{-kt} = 0 \quad \Rightarrow \quad Ce^{-kt} = 1 \quad \Rightarrow \quad e^{-kt} = \frac{1}{C}$

The t-value for which $e^{-kt} = \frac{1}{C}$ is the only possible t-value for the inflection point.

For the t-value for which $e^{-kt} = \frac{1}{C}$, we have that $P(t) = \frac{L}{1+C(\frac{1}{C})} = \frac{L}{1+1} = \frac{L}{2}$